Discrete Time Convolution is Multiplication without Carry

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***Abstract* — In this paper an analysis of discrete-time convolution is performed to prove that the convolution sum is polynomial multiplication without carry, whether the sequences are finite or not, by using several examples to compare the results computed using the existing approaches to the polynomial multiplication approach presented here. In the design and analysis of signals and systems the concept of convolution is very important. While software tools are available for calculating convolution, for proper understanding it is important to learn now to calculate it by hand. To this end, several popular methods are available. The idea that the convolution sum is indeed polynomial multiplication without carry is demonstrated in this paper. The concept is further extended to deconvolution, N-point circular convolution and the Z-transform approach.**

***Keywords* — Convolution, deconvolution, N-point circular convolution, Z-transform.**

I. Introduction

In the design and analysis of signals-whether discrete-time (DT) or continuous-time (CT), the concept of convolution is an indispensable and basic foundation. For CT signals the convolution is usually computed using integrals, while for DT signals, summation is used. The DT convolution usually referred to as the *convolution sum*, for which, there are two types- *N-point circular convolution* (periodic for a period of *N*) and *linear convolution* (basic and from $-\infty$ to $+\infty$). To understand linear convolution, one usually starts by choosing two finite length sequences of 1-dimension and computing the convolution sum by hand. Linear convolution has been described in the major authoritative scholarly textbooks on signals and systems analysis. See for example [1]-[6].

he main solution methods presented in the books are:

- *graphical convolution:* where the graphs of both signals are drawn by hand and manipulated;
- *analytical convolution*: where the solution is obtained by solving an algebraic equation and usually arriving at a closed form solution;
- the *Z-transform approach:* where the Z-transforms of both signals are manipulated to obtain a solution. The analytical solution is more generic because it can obtain a closed-form solution when the lengths of both sequences are unknown. The idea of convolution sum as the multiplication of both signals without carry is not in any of the books.

However, it is acknowledged that it is indispensable to understand analytical convolution because is the only method that can obtain a closed form solution. Furthermore, this paper does not attempt to prove that the multiplication approach presented here is the superior method. Rather it is an alternative method to arrive at the same solution.

[7] was the first paper that showed the analogy of convolution with polynomial multiplication, albeit limited to sequences of finite length. However, this paper attempts to prove and to generalize that it does not matter whether sequences are finite, or infinite or which method is used to calculate, the convolution sum obtained will correspond to the polynomial multiplication of both signals without carry. Therefore, it is valid to calculate and to understand the solution as the polynomial multiplication without carry presented here.

In addition, most popular scholarly textbooks in signals explain equivalence of multiplication in the Z-domain with convolution in the time. Based on this, the Z-transform approach to convolution is done by first obtaining the Z-transforms of both signals, then performing the multiplication of both signals. The convolution sum is taken as the coefficients of the product. [8] shows that for finite length signals, vector multiplication of both signals in the time domain can be directly performed without taking the Z-transforms and it will produce the same result as the Z-transform approach. It will also be shown here that polynomial multiplication can be directly applied to both signals in the time domain without taking the Z-transforms and it will produce the same result as the Z-transform approach.

The remaining parts of the paper is as follows. Section II contains a description of the convolution sum with some examples to demonstrate that the sum obtained by the graphical, analytical and Z-transform approaches are the same as those obtained by direct polynomial multiplication. In Section III it is shown that deconvolution can be performed by long division (as the opposite of multiplication) if one of the original signals is known. Section IV shows the extension of the polynomial multiplication approach to the periodic N-point circular convolution. The paper is concluded in Section V.

II. Convolution Sum Is Multiplication without Carry

In this section, several examples are presented some from the reviewed books to show that the convolution sum is polynomial multiplication without carry. The established methods which are, graphical, Z-transform, and analytical

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solutions will be presented (where necessary, references will be made to solutions in the books with pages indicated). Thereafter the solutions will be compared with that of the polynomial multiplication approach presented here.

A. Non-Finite Sequence Example

Consider Example 2.3.4 from [4] (p. 82), which is as follows. Determine the convolution sum $h[n] = h_1[n] * h_2[n]$ for the signals:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n] \text{ and } h_2[n] = \left(\frac{1}{4}\right)^n u[n].$$

The linear convolution sum is stated as the infinite summation:

$$h[n] = h_1[n] * h_2[n] = \sum_{m=-\infty}^{\infty} h_1[m]\, h_2[n-m]$$

The analytical closed form solution for the above example is:

$$h[n] = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[n]$$

Obtaining $h[n]$ samples for $k = 0, \cdots, 10$ ($h[n] \approx 0$ for $n > 10$) yields the sequence:

$h[n] =$ [1, 0.75, 0.44, 0.234, 0.121, 0.062, 0.031, 0.016, 0.0078, 0.0039, 0.002].

The graph of the solution is shown in Fig. 1.

For the polynomial multiplication solution, finite numeric samples of $h_1[n]$ (for $n = 0, \cdots, 10$) and $h_2[n]$ (for $n = 0, \cdots, 4$) are taken and multiplied without carry. The result is shown in Table I, which clearly shows that the result is the same as the graphical and analytical closed form solutions. This is the first paper where direct polynomial multiplication of samples of the original sequences is used to validate a closed form solution.

TABLE I: $\boldsymbol{h[n]}$ BY VECTOR MULTIPLICATION. MINUTE DIFFERENCES IN LATTER VALUES ARE A RESULT OF APPROXIMATIONS

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
$h_1[n]$	$\hat{1}$	0.5	0.25	0.125	0.0625	0.0313	0.016	0.0078
$h_2[n]$	$\hat{1}$	0.25	0.0625	0.0156	0.0039			
	$\hat{1}$	0.5	0.25	0.125	0.0625	0.0313	0.016	0.0078
		0.25	0.125	0.0625	0.0313	0.0156	0.008	0.0039
			0.0625	0.03125	0.0156	0.0078	0.004	0.002
				0.0156	0.0078	0.0039	0.002	0.001
					0.0039	0.002	1E-03	0.0005
$h[n]$	**$\hat{1}$**	**0.75**	**0.4375**	**0.23435**	**0.1211**	**0.0606**	**0.03**	**0.0151**

n	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$h_1[n]$	0.0039	0.002	0.001				
$h_2[n]$							
	0.0039	0.002	0.001				
	0.002	0.001	0.0005	0.00025			
	0.001	0.0005	0.000244	0.000125	6.25E-05		
	0.0005	0.0002	0.000122	6.08E-05	3.12E-05	1.56E-05	
	0.0002	0.0001	6.08E-05	3.04E-05	1.52E-05	7.80E-06	3.9E-06
$h[n]$	**0.0076**	**0.0038**	**0.001927**	**0.000466**	**0.000109**	**2.34E-05**	**3.9E-06**

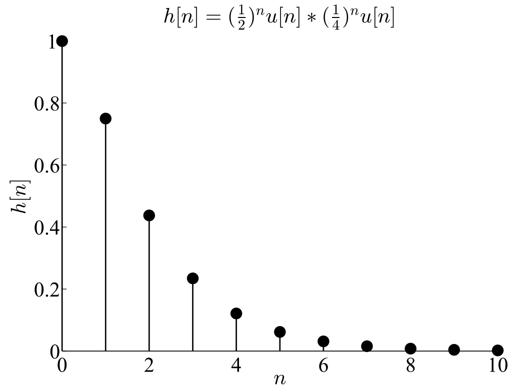

Fig. 1. The convolution sum $\boldsymbol{h[n] = h_1[n] * h_2[n]}$.

B. Finite Sequence Example

Next, consider the finite sequences $x[n] = \{3,2,0,\hat{2},2\}$ and $h[n] = \{2,-\hat{1},1,0,0,2,1\}$. The ˆ indicates the $n = 0$ sample. From the above data $-4 \le m \le 6$ and the analytical solution is given as:

$$y[n] = \sum_{m=-4}^{6} x[m]h[n-m]$$

$$y[-4] = x[-3]h[-1] = 6$$

$$y[-3] = x[-3]h[0] + x[-2]h[-1] = -3 + 4 = 1$$

$$y[-2] = x[-3]h[1] + x[-2]h[0] + x[-1]h[-1] = 3 - 2 + 0 = 1$$

$$y[-1] = x[-3]h[2] + x[-2]h[1] + x[-1]h[0] + x[0]h[-1] = 6$$

$$y[0] = x[-3]h[3] + x[-2]h[2] + x[-1]h[1] + x[0]h[0] + x[1]h[-1] = 0 + 0 + 0 - 2 + 4 = 2$$

$$y[1] = x[-3]h[4] + x[-2]h[3] + x[-1]h[2] + x[0]h[1] + x[1]h[0] = 6 + 0 + 0 + 2 - 2 = 6$$

$$y[2] = x[-3]h[5] + x[-2]h[4] + x[-1]h[3] + x[0]h[2] + x[1]h[1] = 3 + 4 + 0 + 0 + 2 = 9$$

$$y[3] = x[-2]h[5] + x[-1]h[4] + x[0]h[3] + x[1]h[2] = 2 + 0 + 0 + 0 + 0 = 2$$

$$y[4] = x[-1]h[5] + x[0]h[4] + x[1]h[3] = 4$$

$$y[5] = x[0]h[5] + x[1]h[4] = 2 + 4 = 6$$

$$y[6] = x[1]h[5] = 2$$

$$y[n] = \{6,1,1,6,\hat{2},6,9,2,4,6,2\}$$

The solution obtained by direct multiplication is as follows:

n	-4	-3	-2	-1	0	1	2	3	4	5	6
$h[n]$				2	$-\hat{1}$	1	0	0	2	1	
$x[n]$		3	2	0	$\hat{2}$	2					
$x[n]h[-$	6	4	0	4	4						
$x[n]h[0$		-3	-2	0	-2	-2					
$x[n]h[1$			3	2	0	2	2				
$x[n]h[2$				0	0	0	0	0			
$x[n]h[3$					0	0	0	0	0		
$x[n]h[4$						6	4	0	4	4	
$x[n]h[5$							3	2	0	2	2
$y[n]$	6	1	1	6	$\hat{2}$	6	9	2	4	6	2

Observe that each $y[n]$ of the analytical solution corresponds to each column sum in the above solution.

Also note that the multiplication is started from right to left in this case rather than the usual multiplication from left to right (as in the first example) and in [7]. Both methods give the same result, however, with this method the $n = 0$ sample of both convoluted signals are visibly aligned with that of the convolution sum obtained.

In addition, in [1] Example 9.9 (pp. 594-595), a *sliding tape method* is presented, intended as an alternative to graphical convolution but has not been as popular. The table below presents the polynomial multiplication solution to [1] Example 9.9. In the solution, each summed column corresponds to the $c[k]$ terms of the sliding tape method.

$f[k]$	$\hat{0}$	1	2	3	4	5					
$g[k]$	$\hat{1}$	1	1	1	1	1					
	0	1	2	3	4	5					
		0	1	2	3	4	5				
			0	1	2	3	4	5			
				0	1	2	3	4	5		
					0	1	2	3	4	5	
						0	1	2	3	4	5
$c[k]$	$\hat{0}$	1	3	6	10	15	15	14	12	9	5

Clearly, this is the same solution as that obtained from the sliding tape method.

In addition, a *table of convolution sums* is presented [1] (p. 590) to aid in the calculation of analytical convolution.

C. Z-transform Example

In [8], it is shown that if a closed form solution is not necessary and the sequences are finite, it is not necessary to apply the Z-transform before vector multiplication in order to simplify the computation of the convolution sum. There vector multiplication was done directly, and the same solution was obtained. Here the polynomial multiplication (rather than vector multiplication) will be applied directly to obtain the same solution as the Z-transform approach. To this end, consider the following example: compute the convolution of

$$x_1[n] = \{-2, \hat{1}, 3, 2\}; \quad x_2[n] = \begin{cases} 1, & 0 \leq n \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

The Z-transform solution approach is as follows:

Step 1: calculate Z-transform of both signals

$$X_1(z) = -2z + 1 + 3z^{-1} + 2z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3}$$

Step 2: multiply both transforms to obtain:

$$X(z) = X_1(z)X_2(z)$$

as follows:

$-2z$	1	$3z^{-1}$	$+2z^{-2}$			
	1	$+z^{-1}$	$+z^{-2}$	$+z^{-3}$		
$-2z$	-2	$-2z^{-1}$	$-2z^{-2}$			
	1	$+z^{-1}$	$+z^{-2}$	$+z^{-3}$		
		$+3z^{-1}$	$+3z^{-2}$	$+3z^{-3}$	$+3z^{-4}$	
			$+2z^{-2}$	$+2z^{-3}$	$+2z^{-4}$	$+2z^{-5}$
$-2z$	-1	$+2z^{-1}$	$+4z^{-2}$	$+6z^{-3}$	$+5z^{-4}$	$+2z^{-5}$

Step 3: obtain the convolution sum as the coefficients of $X(z)$:

$$x[n] = [-2, -\hat{1}, 2, 4, 6, 5, 2]$$

The direct polynomial multiplication solution to the example is presented below.

$x_1[n]$	-2	$\hat{1}$	3	2			
$x_2[n]$	$\hat{1}$	1	1	1			
	-2	-2	-2	-2			
		1	1	1	1		
			3	3	3	3	
				2	2	2	2
$x[n]$	-2	$-\hat{1}$	2	4	6	5	2

The comparison of the solutions obtained using the various approaches and the complexities involved is evident in the examples seen in this section.

III. Deconvolution

Deconvolution is the process of recovering one of the original signals of a convolution sum, given the sum and one of the original signals. It is useful in such problems as system identification in the sense that, knowing the output (convolution sum) of a system $y[n]$ and its impulse response $h[n]$, one may determine the input $x[n]$. A few textbooks e.g. [1, p. 615], [3, pp. 773-775] and [4, pp. 349-363], contain a basic overview of deconvolution. Homomorphic deconvolution is also described in [4].

Deconvolution may be approached as a long division problem (the opposite of multiplication). The long division approach in computing inverse Z-transforms is presented in [1]-[6].

Consider the previous Z-transform example (II *C*). Given $x_2[n] = [\hat{1}, 1,1,1]$ and $x[n]$, one may obtain $x_1[n]$ by long division as follows:

$$
\begin{array}{r|l}
 & -2 \quad 1 \quad 3 \quad 2 \\
\hline
\hat{1}\,1\,1\,1 & -2 \;\; -\hat{1} \;\; 2 \;\; 4 \;\; 6 \;\; 5 \;\; 2 \\
 & -2 \;\; -2 \;\; -2 \;\; -2 \\
\hline
 & 1 \;\; 4 \;\; 6 \;\; 6 \\
 & 1 \;\; 1 \;\; 1 \;\; 1 \\
\hline
 & 3 \;\; 5 \;\; 5 \;\; 5 \\
 & 3 \;\; 3 \;\; 3 \;\; 3 \\
\hline
 & 2 \;\; 2 \;\; 2 \;\; 2 \\
 & 2 \;\; 2 \;\; 2 \;\; 2 \\
\hline
 & 0 \;\; 0 \;\; 0 \;\; 0 \\
\hline
\end{array}
$$

In the above example one of the sequences is causal. Consider the finite sequence example (II *B*) in which $h[n]$ and $y[n]$ are non-causal. Given $h[n] = [2, -\hat{1}, 1,0,0,2,1]$ and $y[n]$, $x[n]$ is obtained by long division as follows:

$$
\begin{array}{r|l}
 & 3 \;\; 2 \;\; 0 \;\; 2 \;\; 2 \\
\hline
2-\hat{1}\,10021 & 6 \;\; 1 \;\; 1 \;\; 6 \;\; \hat{2} \;\; 6 \;\; 9 \;\; 2 \;\; 4 \;\; 6 \;\; 2 \\
 & 6 \;\; -3 \;\; 3 \;\; 0 \;\; 0 \;\; 6 \;\; 3 \\
\hline
 & 4 \;\; -2 \;\; 6 \;\; 2 \;\; 0 \;\; 6 \;\; 2 \\
 & 4 \;\; -2 \;\; 2 \;\; 0 \;\; 0 \;\; 4 \;\; 2 \\
\hline
 & 0 \;\; 4 \;\; 2 \;\; 0 \;\; 2 \;\; 0 \;\; 4 \\
 & 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \\
\hline
 & 4 \;\; 2 \;\; 0 \;\; 0 \;\; 0 \;\; 4 \;\; 6 \\
 & 4 \;\; -2 \;\; 2 \;\; 0 \;\; 0 \;\; 4 \;\; 2 \\
\hline
 & 4 \;\; -2 \;\; 0 \;\; 0 \;\; 0 \;\; 4 \;\; 2 \\
 & 4 \;\; -2 \;\; 0 \;\; 0 \;\; 0 \;\; 4 \;\; 2 \\
\hline
\end{array}
$$

To determine the $n = 0$ point in the dividend $x[n]$ subtract the number of $n < 0$ points of the numerator $y[n]$ from that of the denominator $h[n]$. The absolute value of the difference is the number of $n < 0$ points in $x[n]$. In this case $x[n]$ has $4 - 1 = 3$ $n < 0$ points, therefore $x[n] = [3,2,0,\hat{2},2]$.

IV. Circular (Periodic) Convolution

In [8], a vector multiplication approach was developed for the calculation of circular (periodic) and it was shown that the solution obtained is equivalent to the methods presented in the reviewed standard textbooks [1, p. 349-351, 651-652], [3, pp. 399-426], [4, pp. 471-474], and [5, pp. 676-687]. In this section the polynomial multiplication approach will be extended to circular convolution.

The development is started by considering Exercise 7.8 in [4, p. 503], stated as follows. Calculate the circular convolution"

$$y[n] = x_1[n] \circledast x_2[n]$$

of the sequences:

$$x_1[n] = [\hat{1}, 2, 3, 1], x_2[n] = [\hat{4}, 3, 2, 2]$$

The above data shows that $N = 4$. The analytical solution is obtained using the time-domain formula, equation (7.2.39) in [4] as follows:

$$y[n] = \sum_{k=0}^{3} x_1[k]x_2\left[((n-k))_N\right],\ \ 0 \le n \le 3$$

$$
\begin{aligned}
y[0] &= x_1[0]x_2\left[((0))_4\right] + x_1[1]x_2\left[((-1))_4\right] + x_1[2]x_2\left[((-2))_4\right] \\
&\quad + x_1[3]x_2\left[((-3))_4\right] \\
&= 1\times 4 + 2\times 2 + 3\times 2 + 1\times 3 = 17
\end{aligned}
$$

$$
\begin{aligned}
y[1] &= x_2\left[((1))_4\right] + 2x_2\left[((0))_4\right] + 3x_2\left[((-1))_4\right] + x_2\left[((-2))_4\right] \\
&= 1\times 3 + 2\times 4 + 3\times 2 + 1\times 2 = 19
\end{aligned}
$$

$$
\begin{aligned}
y[2] &= x_2\left[((2))_4\right] + 2x_2\left[((1))_4\right] + 3x_2\left[((0))_4\right] + x_2\left[((-1))_4\right] \\
&= 1\times 2 + 2\times 3 + 3\times 4 + 1\times 2 = 22
\end{aligned}
$$

$$
\begin{aligned}
y[3] &= x_2\left[((3))_4\right] + 2x_2\left[((2))_4\right] + 3x_2\left[((1))_4\right] + x_2\left[((0))_4\right] \\
&= 1\times 2 + 2\times 2 + 3\times 3 + 1\times 4 = 19
\end{aligned}
$$

$$y[n] = \{\widehat{17}, 19, 22, 19\}$$

The part of determining the circular shifting $x_2\left[((n-k))_N\right]$, $0 \le n \le 3$ is not shown in the above solution, so, one may appreciate the added complexity in comparison to analytical linear convolution.

Circular shifting is illustrated in [4] (p. 473) using a series of circular discs. It has been shown in [8] that one may represent the required circular shifting by using one disc as shown in Fig. 2.

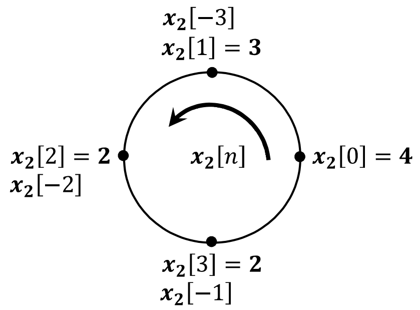

Fig. 2. From [8], circular shifting requires one clockwise pass from $x_2[3]$ to $x_2[0]$ followed by another clockwise pass from $x_2[-1]$ to $x_2[-3]$, thus the sequence $x_2\left[((n-k))_N\right]$ is 2,2,3,4,2,2,3.

From the sequence, the polynomial multiplication solution is:

$n-k$	3	2	1	0	-1	-2	-3	
$x_2\left[((n-k))_N\right]$	2	2	3	4	2	2	3	
$x_1[n]$	1	2	3	1				
	2	4	9	4				$y[3] = 19$
$x_1[n] \rightarrow$		2	6	12	2			$y[2] = 22$
$x_1[n] \rightarrow$			3	8	6	2		$y[1] = 19$
$x_1[n] \rightarrow$				4	4	6	3	$y[0] = 17$

Observe that direct multiplication is not applied here like the linear convolution, rather it is similar to the sliding tape method in [1] for linear convolution. At each stage the elements of $x_2\left[((n-k))_N\right]$ and $x_1[n]$ are multiplied just as a dot product, then the row sum is taken. Then $x_1[n]$ is slid right one step for the next stage. The tabular presentation here is different from the approaches presented in the books and in [7].

It was also shown in [8] that it does not matter which direction the circular shifting is made-an *anticlockwise* pass from $x_2[-3]$ to $x_2[0]$ can also be made, followed by another anticlockwise pass from $x_2[1]$ to $x_2[3]$. In this case $x_1[n]$ is also taken anticlockwise $(x_1[-n])$. The polynomial multiplication follows as:

$n-k$	3	2	1	0	-1	-2	-3	
$x_2\left[((n-k))_N\right]$	3	2	2	4	3	2	2	
$x_1[n]$	1	3	2	1				
	3	6	4	4				$y[0] = 17$
$x_1[n] \rightarrow$		2	6	8	3			$y[1] = 19$
$x_1[n] \rightarrow$			2	12	6	2		$y[2] = 22$
$x_1[n] \rightarrow$				4	9	4	2	$y[3] = 19$

Clearly, the solutions from all the approaches are the same.

V. Conclusion

The aim of this paper is to show that the convolution sum is equivalent to polynomial multiplication without carry, whether the sequences are finite or not, therefore, the problem can be reduced to a direct multiplication problem.

Also, following [8], it is here reiterated that for finite sequences, polynomial multiplication can be done directly without the Z-transform to obtain the same convolution sum as the Z-transform approach. Thereby reducing complexity because it involves much fewer steps (only the last step of the Z-transform method).

The polynomial multiplication approach presented is an additional tool that can be adopted for practice.

As a part of future work, one may consider investigating simplified methods of solving CT convolution problems by hand. Topics such as circular correlation and circular autocorrelation can also be considered. Another topic is to investigate the computational complexities of the different approaches.

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