Analysis of Lossy Transmission Lines Terminated by Schottky Diode Circuit

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Abstract—In the present paper we consider a lossy transmission line terminated by a circuit corresponding to a Schottky diode. On the base of Kirchhoff's law boundary conditions are derived. Then a mixed problem for the lossy transmission line system is formulated. We reduce the mixed problem for the hyperbolic transmission line system to an initial value problem for a system of differential equations with delays on the boundary. We prove existence-uniqueness theorem for oscillatory solution. The paper ends with numerical example with real values of the Schottky diode parameters.

Index Terms—Fixed Point Method; Lossy Transmission Line; Oscillatory Solution; Schottky Diode.

I. INTRODUCTION

The main purpose of the present paper is to investigate the propagation of signals along a lossy transmission line terminated by a circuit corresponding to a Schottky-barrier diode. In this case the circuit consists of parallel connected nonlinear RC -loads and in series connected R -load (cf. [1] - [4]). The nonlinear characteristics of RC -loads generate nonlinearities in the differential equations on the boundary with delays. Our considerations are under the Heaviside condition $L/R = G/C$, where L, R, G, C are specific parameters of the transmission line (L is per unit-length inductance, R – per unit-length resistance, G – per unit-length conductance, C – per unit-length capacitance) and Λ is its length, $v = 1/\sqrt{LC}$ is the speed of propagation and $T = \Lambda/(1/\sqrt{LC}) = \Lambda\sqrt{LC}$ is the time delay.

The paper consists of seven sections. First we formulate the mixed problem for the hyperbolic system describing a lossy transmission line and derive the boundary conditions (on the base of the Kirchhoff's law) generated by Schottky diode circuit. We emphasize that in this case the RC current function cannot be eliminated but we extend our method (cf. [5]–[8]) and consider three equations introducing additional current (voltage). Various applications of such circuits are given in [9] - [24]. In Section 2 we transform the hyperbolic system in a diagonal form. In Section 3 we reduce the mixed problem to an initial value problem on the boundary. We show that natural solutions are oscillatory ones. In Section 4 we analyze nonlinearities and give an operator presentation of the oscillatory problem. In other words, we introduce an operator whose fixed point is an oscillatory solution of the

initial value problem on the boundary. Section 5 contains a preliminary assertion. In Section 6 we prove the main result – the existence-uniqueness theorem of an oscillatory solution of differential system with delays. Finally, in Section 7, we give a numerical example which demonstrates the advantages of our method.

Here we consider a lossy transmission line described by the system

$$\begin{aligned} \frac{\partial u(x,t)}{\partial x} + L \frac{\partial i(x,t)}{\partial t} + Ri(x,t) &= 0, \\ \frac{\partial i(x,t)}{\partial x} + C \frac{\partial u(x,t)}{\partial t} + Gu(x,t) &= 0 \end{aligned} \quad (1)$$

which can be rewritten in the matrix form

$$\frac{\partial U(x,t)}{\partial t} + A \frac{\partial U(x,t)}{\partial x} + A_1 U = \bar{0} \quad (2)$$

where

$$U = \begin{bmatrix} u(x,t) \\ i(x,t) \end{bmatrix}, \quad \frac{\partial U}{\partial t} = \begin{bmatrix} \frac{\partial u(x,t)}{\partial t} \\ \frac{\partial i(x,t)}{\partial t} \end{bmatrix}, \quad \frac{\partial U}{\partial x} = \begin{bmatrix} \frac{\partial u(x,t)}{\partial x} \\ \frac{\partial i(x,t)}{\partial x} \end{bmatrix},$$

$$A = \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} G/C & 0 \\ 0 & R/L \end{bmatrix}.$$

Formulate mixed problem for (1): to find a solution $u(x,t), i(x,t)$ for $(x,t) \in \Pi = \{(x,t) \in \mathbb{R}^2 : 0 \leq x \leq \Lambda, t \geq 0\}$, satisfying the initial conditions

$$u(x,0) = u_0(x), i(x,0) = i_0(x) \quad \text{for } x \in [0, \Lambda] \quad (3)$$

and boundary conditions derived bellow.

The Schottky barrier diode has an equivalent circuit shown on Fig. 1. Here C_0 and R_1 are nonlinear capacitance and resistance, while R_2 – linear resistance.

For the capacitance we choose (cf. Maas [1])

$$Q_0(u) = -C_{j0} \Phi(1 - (u/\Phi))^{1-\gamma} / (1-\gamma)$$

and consequently $C_0(u) = \frac{dQ_0(u)}{du} = C_{j0}(1 - (u/\Phi))^{-\gamma}$, where constants C_{j0}, Φ, γ are explained finally in the numerical example.

The current of the capacitance is

$$i_{C_0} = \frac{dQ_0(u_{C_0})}{du} \frac{du_{C_0}}{dt} = C_{j0} (1 - (u_{C_0}/\Phi))^{-\gamma} \frac{du_{C_0}}{dt}.$$

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Fig. 1. Lossy transmission line loaded by Schottky diode circuit

If $|u_{C_0}| \leq \phi_0 e^{\mu_0} < \Phi$ then $dC_0(u)/du$ has strictly positive lower bounds and upper bound. Indeed,

$$\frac{dC_0(u_{C_0})}{du_{C_0}} = C_{j0} \gamma (1 - (u_{C_0}/\Phi))^{-\gamma-1} / \Phi > 0$$

and therefore

$$C_{j0} (1 + (\phi_0 e^{\mu_0} / \Phi))^{-\gamma} \leq C_0(u_{C_0}) \leq C_{j0} (1 - (\phi_0 e^{\mu_0} / \Phi))^{-\gamma}.$$

The current of the resistance R_1 is

$$i_{R_1}(u_{R_1}) = I_{sat} (e^{\alpha u_{R_1}} - 1).$$

Obviously both elements have nonlinear characteristics that generate similar nonlinearities in the boundary conditions.

Indeed, applying the Kirchhoff's law for the right end of the line we have $i_{C_0} + i_{R_1} = i(\Lambda, t)$. But $u_{C_0} = u_{R_1}$ and $u_{R_1} + u_{R_2} = u(\Lambda, t) \Leftrightarrow u_{C_0} + u_{R_2} = u(\Lambda, t)$. Therefore

$$C_{j0} (1 - (u/\Phi))^{-\gamma} \frac{du_{C_0}}{dt} + I_{sat} (e^{\alpha u_{C_0}} - 1) = i(\Lambda, t);$$

$$u_{C_0} + R_2 i_{R_2} = u(\Lambda, t).$$

But $i_{R_2} = i(x, t)$. In this manner we obtain the boundary conditions (cf. Fig. 1) for $x = 0$

$$E(t) - u(0, t) - R_0 i(0, t) = 0, t \geq 0 \quad (4)$$

and in view of $i_{R_2}(\Lambda, t) = i(\Lambda, t)$ for $x = \Lambda$

$$u_{C_0} + R_2 i(\Lambda, t) = u(\Lambda, t),$$

$$C_{j0} (1 - (u/\Phi))^{-\gamma} \frac{du_{C_0}}{dt} + I_{sat} (e^{\alpha u_{C_0}} - 1) = i(\Lambda, t). \quad (5)$$

II. TRANSFORMATION OF THE HYPERBOLIC SYSTEM IN A DIAGONAL FORM

Consider the system (2). The eigenvalues of the matrix A are $\lambda_1 = 1/\sqrt{LC}$, $\lambda_2 = -1/\sqrt{LC}$ and corresponding eigenvectors are $(\xi_1^{(1)}, \xi_2^{(1)}) = (\sqrt{C}, \sqrt{L})$, $(\xi_1^{(2)}, \xi_2^{(2)}) = (-\sqrt{C}, \sqrt{L})$.

The matrix H formed by the eigenvectors is $H = \begin{bmatrix} \sqrt{C} & \sqrt{L} \\ -\sqrt{C} & \sqrt{L} \end{bmatrix}$

and its inverse is $H^{-1} = \begin{bmatrix} 1/(2\sqrt{C}) & -1/(2\sqrt{C}) \\ 1/(2\sqrt{L}) & 1/(2\sqrt{L}) \end{bmatrix}$.

It is known that $A^{\text{can}} = HAH^{-1}$, where

$$A^{\text{can}} = \begin{bmatrix} 1/\sqrt{LC} & 0 \\ 0 & -1/\sqrt{LC} \end{bmatrix}.$$

Introduce new variables $Z = \begin{bmatrix} V(x, t) \\ I(x, t) \end{bmatrix}$, where $Z = HU$

and $U = H^{-1}Z$ or

$$\begin{cases} V(x, t) = \sqrt{C} u(x, t) + \sqrt{L} i(x, t) \\ I(x, t) = -\sqrt{C} u(x, t) + \sqrt{L} i(x, t) \end{cases}$$

and

$$\begin{cases} u(x, t) = (V(x, t) - I(x, t))/(2\sqrt{C}) \\ i(x, t) = (V(x, t) + I(x, t))/(2\sqrt{L}). \end{cases}$$

Substituting $U = H^{-1}Z$ in (2) we obtain

$$\frac{\partial(H^{-1}Z)}{\partial t} + A \frac{\partial(H^{-1}Z)}{\partial x} + A_1(H^{-1}Z) = \bar{0},$$

$$H^{-1} \frac{\partial Z}{\partial t} + AH^{-1} \frac{\partial Z}{\partial x} + A_1(H^{-1}Z) = \bar{0}.$$

Then multiplying the last matrix equality by H from the left we obtain

$$\frac{\partial Z}{\partial t} + HAH^{-1} \frac{\partial Z}{\partial x} + (HA_1H^{-1})Z = \bar{0}$$

where $HA_1H^{-1} = \frac{1}{2} \begin{bmatrix} G/C + R/L & -G/C + R/L \\ -G/C + R/L & G/C + R/L \end{bmatrix}$.

We assume that Heaviside condition $G/C = R/L$ is fulfilled. Then $HA_1H^{-1} = \begin{bmatrix} R/L & 0 \\ 0 & R/L \end{bmatrix}$ and the hyperbolic

system becomes

$$\frac{\partial V}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial V}{\partial x} + \frac{R}{L} V = 0, \quad \frac{\partial I}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial I}{\partial x} + \frac{R}{L} I = 0.$$

Let us put $V(x, t) = e^{-Rt/L} W(x, t)$, $I(x, t) = e^{-Rt/L} J(x, t)$.

Substituting in the last system we reach

$$\frac{\partial W(x, t)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial W(x, t)}{\partial x} = 0, \quad \frac{\partial J(x, t)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial J(x, t)}{\partial x} = 0.$$

Finally, the transformation formulas are

$$\begin{cases} W(x, t) = e^{Rt/L} \sqrt{C} u(x, t) + e^{Rt/L} \sqrt{L} i(x, t) \\ J(x, t) = -e^{Rt/L} \sqrt{C} u(x, t) + e^{Rt/L} \sqrt{L} i(x, t) \end{cases}$$

and

$$\begin{cases} u(x, t) = e^{-Rt/L} W(x, t)/(2\sqrt{C}) - e^{-Rt/L} J(x, t)/(2\sqrt{C}) \\ i(x, t) = e^{-Rt/L} W(x, t)/(2\sqrt{L}) + e^{-Rt/L} J(x, t)/(2\sqrt{L}). \end{cases}$$

Prior to formulate an operator corresponding to the mixed problem we consider the Cauchy problems for the characteristics (Here we mean the characteristics of the hyperbolic system):

$$\frac{dx_W(\tau)}{d\tau} = 1/\sqrt{LC}, \quad x_W(t) = x; \quad \frac{dx_J(\tau)}{d\tau} = -1/\sqrt{LC}, \quad x_J(t) = x \quad (6)$$

for each $(x, t) \in \Pi$. Here the characteristic functions

$\lambda_W = 1/\sqrt{LC} > 0$ and $\lambda_J = -1/\sqrt{LC} < 0$ are continuous ones.

We need the following

Lemma 1. For every $(x, t) \in \text{int } \Pi$ the initial value

problem

$$\frac{dx_W(\tau)}{d\tau} = 1/\sqrt{LC}, x_W(t) = x; \left(\frac{dx_J(\tau)}{d\tau} = -1/\sqrt{LC}, x_J(t) = x \right)$$

has a unique solution belonging to Π .

III. REDUCING THE MIXED PROBLEM TO AN INITIAL VALUE PROBLEM ON THE BOUNDARY

Here we formulate the mixed problem: to find a solution $W(x, t), J(x, t)$ satisfying the system

$$\frac{\partial W(x, t)}{\partial t} + \frac{1}{\sqrt{LC}} \frac{\partial W(x, t)}{\partial x} = 0, \frac{\partial J(x, t)}{\partial t} - \frac{1}{\sqrt{LC}} \frac{\partial J(x, t)}{\partial x} = 0, \quad (7)$$

initial conditions

$$\begin{cases} W(x, 0) = \sqrt{C} u(x, 0) + \sqrt{L} i(x, 0) = \sqrt{C} u_0(x) + \sqrt{L} i_0(x) \equiv W_0(x) \\ J(x, t) = -\sqrt{C} u(x, 0) + \sqrt{L} i(x, 0) = -\sqrt{C} u_0(x) + \sqrt{L} i_0(x) \equiv J_0(x) \end{cases}, \quad (8)$$

and boundary conditions for $x = 0$:

$$E(t) - u(0, t) - R_0 i(0, t) = 0, t \geq 0$$

and for $x = \Lambda$:

$$C_{j0} (1 - (u/\Phi))^{-\gamma} \frac{du_{C_0}}{dt} + I_{sat} (e^{\alpha u_{C_0}} - 1) = i(\Lambda, t);$$

$$u_{C_0}(t) + R_2 i_{R_2}(\Lambda, t) = u(\Lambda, t)$$

which can be transformed in view of

$$\begin{cases} u(0, t) = e^{-Rt/L} (W(0, t) - J(0, t)) / (2\sqrt{C}) \\ i(0, t) = e^{-Rt/L} (W(0, t) + J(0, t)) / (2\sqrt{L}), \\ u(\Lambda, t) = e^{-Rt/L} (W(\Lambda, t) - J(\Lambda, t)) / (2\sqrt{C}) \\ i(\Lambda, t) = e^{-Rt/L} (W(\Lambda, t) + J(\Lambda, t)) / (2\sqrt{L}) \end{cases}$$

in the form

$$\begin{aligned} E(t) - e^{-Rt/L} (W(0, t) + J(0, t)) / (2\sqrt{C}) \\ - e^{-Rt/L} (R_0 W(0, t) - R J(0, t)) / (2\sqrt{L}) = 0, t \geq 0; \\ u_{C_0} + R_2 e^{-Rt/L} (W(\Lambda, t) + J(\Lambda, t)) / (2\sqrt{L}) \\ = e^{-Rt/L} (W(\Lambda, t) - J(\Lambda, t)) / (2\sqrt{C}); \\ \frac{C_{j0}}{(1 - (u_{C_0}(t)/\Phi))^\gamma} \frac{du_{C_0}(t)}{dt} + I_{sat} (e^{\alpha u_{C_0}(t)} - 1) \\ = e^{-Rt/L} (W(\Lambda, t) + J(\Lambda, t)) / (2\sqrt{L}). \end{aligned}$$

Integrating the first equation of (7) along the first characteristic we obtain

$$\begin{aligned} \int_t^{t+T} \left(\frac{\partial W}{\partial x} \frac{dx_W}{ds} + \frac{\partial W}{\partial s} \right) ds = \int_t^{t+T} \frac{dW(x_W(s), s)}{ds} ds = \\ = W(\Lambda, t+T) - W(0, t) = 0. \end{aligned}$$

In a similar way integrating along the second characteristic from $(0, t+T)$ to (Λ, t) we obtain

$$\int_{t+T}^t \left(\frac{\partial J}{\partial x} \frac{dx_J}{ds} + \frac{\partial J}{\partial s} \right) ds = \int_{t+T}^t \frac{dJ(x_J(s), s)}{ds} ds = J(\Lambda, t) - J(0, t+T) = 0.$$

Therefore $W(\Lambda, t) = W(0, t-T)$, $J(\Lambda, t-T) = J(0, t)$.

We assume that $W(0, t) = W(t)$; $J(\Lambda, t) = J(t)$ are

unknown functions. Then the boundary conditions become

$$\begin{aligned} E(t) - e^{-Rt/L} (W(t) - J(t-T)) / (2\sqrt{C}) \\ - e^{-Rt/L} (R_0 W(t) + R J(t-T)) / (2\sqrt{L}) = 0; \end{aligned}$$

$$\begin{aligned} u_{C_0} + e^{-Rt/L} R_2 (W(t-T) + J(t)) / (2\sqrt{L}) \\ = e^{-Rt/L} (W(t-T) - J(t)) / (2\sqrt{C}); \end{aligned}$$

$$\begin{aligned} C_{j0} (1 - (u_{C_0}/\Phi))^{-\gamma} \frac{du_{C_0}(t)}{dt} + I_{sat} (e^{\alpha u_{C_0}} - 1) \\ = e^{-Rt/L} (W(t-T) + J(t)) / (2\sqrt{L}). \end{aligned}$$

It is proved in [8] that the above relations are equivalent to the mixed problem for the hyperbolic system. Using the denotation $Z_0 = \sqrt{L/C}$ we present the last equations in a fixed point form and we are able to formulate the main problem: to find an oscillatory solution $(W(t), J(t), u_{C_0}(t))$ of the system

$$\begin{aligned} W(t) &= \frac{2e^{Rt/L} E(t) \sqrt{L}}{Z_0 + R_0} + \frac{Z_0 - R_0}{Z_0 + R_0} J(t-T), \\ J(t) &= -\frac{2e^{Rt/L} u_{C_0}(t) \sqrt{L}}{Z_0 + R_2} + \frac{Z_0 - R_2}{Z_0 + R_2} W(t-T), \\ \frac{du_{C_0}(t)}{dt} &= -\left(1 - (u_{C_0}(t)/\Phi)\right)^\gamma / C_{j0} \\ &\cdot \left(I_{sat} (e^{\alpha u_{C_0}} - 1) - (W(t-T) + J(t)) e^{-Rt/L} / (2\sqrt{L}) \right), \\ W(t) &= W_0(t), \quad J(t) = J_0(t), \quad t \in [0, T] \end{aligned} \quad (9)$$

with prescribed initial functions on $[0, T]$ obtained from (8) as in [8]. The constants $\alpha, \Phi, \gamma, I_{sat}, C_{j0}$ will be commented in the numerical example at the very end of the paper. The right-hand sides of (9) show that natural solutions are oscillatory ones. Periodic solutions do not exist in view of $e^{-Rt/L}$.

IV. OPERATOR PRESENTATION OF THE OSCILLATORY PROBLEM

Here we introduce an operator acting on a uniform space of oscillatory functions [8]. Its fixed points are oscillatory solutions of the problem in question (9).

We look for a solution of (9) with advanced prescribed zeroes on $[t_0 = T, \infty)$. First we define the sets of zeros. The prescribed initial functions $W_0(t), J_0(t), t \in [0, T]$ have zeros $\tau_0 = 0 < \tau_1 < \dots < \tau_n = T$. Then we translate this picture on $[T, 2T], [2T, 3T], \dots$, that is,

$$t_0 = T < t_1 = \tau_1 + T < \dots < t_n + T = 2T,$$

$$2T = t_{n+1} = t_n + T < t_{n+2} = t_n + T < \dots < t_{n+n} + T = 3T, \dots$$

It follows that $t_k - T = t_{k-n}$ for every k .

Denote by $T_0 = \max\{\tau_{k+1} - \tau_k : k = 0, 1, \dots, n-1\}$.

Obviously $S = \{t_k\}_{k=0}^\infty$ is a strictly increasing sequence

for which $\lim_{k \rightarrow \infty} t_k = \infty$.

By $C[t_0, \infty)$ we denote the set of continuous and bounded functions on interval $[t_0, \infty)$. The initial functions $W_0(\cdot), J_0(\cdot) \in C[0, T]$ satisfy the inequalities:

$$|W_0(t)| \leq W_0 e^{\mu(t-\tau_k)}, \quad |J_0(t)| \leq J_0 e^{\mu(t-\tau_k)}, \quad t \in [\tau_k, \tau_{k+1}],$$

$$(k = 0, 1, 2, \dots, n-1).$$

Introduce the sets

$$M_W = \{W(\cdot) \in C[t_0, \infty) : W(t_k) = 0 \wedge |W(t)| \leq W_0 e^{\mu(t-t_k)}\},$$

$$M_J = \{J(\cdot) \in C[t_0, \infty) : J(t_k) = 0 \wedge |J(t)| \leq J_0 e^{\mu(t-t_k)}\},$$

$$M_0 = \{u_{C_0}(\cdot) \in C[t_0, \infty) : u_{C_0}(t_k) = 0 \wedge |u_{C_0}(t)| \leq \phi_0 e^{(\mu - (R/L))(t-t_k)}\}$$

for $t \in [t_k, t_{k+1}]$.

Assumption (Φ): $\phi_0 e^{\mu T_0} < \Phi$.

Here $W_0, J_0, \phi_0, \mu, \mu T_0 = \mu_0$ are positive constants.

Introduce the following family of pseudo-metrics

$$\rho_k(V, \bar{V}) = \max_{t \in [t_k, t_{k+1}]} \{e^{-\mu(t-t_k)} |V(t) - \bar{V}(t)|\},$$

$$\rho_k(J, \bar{J}) = \max_{t \in [t_k, t_{k+1}]} \{e^{-\mu(t-t_k)} |J(t) - \bar{J}(t)|\},$$

$$\rho_k(u_{C_0}, \bar{u}_{C_0}) = \max_{t \in [t_k, t_{k+1}]} \{e^{-(\mu - (R/L))(t-t_k)} |u_{C_0}(t) - \bar{u}_{C_0}(t)|\}.$$

The set $(X, A) = M_W \times M_J \times M_0$ turns out into a complete uniform space (cf. [25]) with respect to the countable family of pseudo-metrics ($k = 0, 1, 2, \dots$)

$$\rho_k((W, J, u_{C_0}), (\bar{W}, \bar{J}, \bar{u}_{C_0}))$$

$$= \max\{\rho_k(W, \bar{W}), \rho_k(J, \bar{J}), \rho_k(u_{C_0}, \bar{u}_{C_0})\}.$$

Here the index set is $A = \{0, 1, 2, \dots\}$.

A map $B: (X, A) \rightarrow (X, A)$ is called Φ -contractive iff

$$\rho_k(B(W, J, u_{C_0}), B(\bar{W}, \bar{J}, \bar{u}_{C_0})) \leq L_0 \rho_{j(k)}(B(W, J, u_{C_0}), B(\bar{W}, \bar{J}, \bar{u}_{C_0}))$$

($k = 0, 1, 2, \dots$), $L_0 < 1$, where the map $j: A \times X \times X \rightarrow A$ is defined as follows: $j^0(k) = k$,

$$j(k) = \begin{cases} k, & \text{if } \max\{\rho_{k-n}(W, \bar{W}), \rho_{k-n}(J, \bar{J}), \rho_k(J, \bar{J}), \rho_k(u_{C_0}, \bar{u}_{C_0})\} \\ & \text{is attended for } k \\ k-n, & \text{if } \max\{\rho_{k-n}(W, \bar{W}), \rho_{k-n}(J, \bar{J}), \rho_k(J, \bar{J}), \rho_k(u_{C_0}, \bar{u}_{C_0})\} \\ & \text{is attended for } k-n. \end{cases}$$

and $j^m(k) = j(j^{m-1}(k))$. Obviously after every translation $t-T$ moves every interval $[t_k, t_{k+1}]$ to the left and after finite number of iterations its image turns out to the left of the initial point or does not depend on m . Consequently

$$\rho_{j^m(k)}((W, J, u_{C_0}), (\bar{W}, \bar{J}, \bar{u}_{C_0})) \leq Q < \infty \quad (m = 0, 1, 2, 3, \dots),$$

that is, the uniform space is j -bounded (cf. [25], [26]). This implies a uniqueness of the fixed point of the Φ -contractive map.

Prior to define a operator B we consider the space $C[0, \infty)$ with zeros $\{\tau_k\}_{k=0}^n \cup \{t_k\}_{k=0}^\infty$ which coincide with $W_0(\cdot)$ on $[0, T]$. In view of the definition of $\{\tau_k\}_{k=0}^n \cup \{t_k\}_{k=0}^\infty$ every translated to the right function (with length T) coincides with some function $\tilde{W}(\cdot) \in C[t_0, \infty)$. In a

similar way we proceed for $\tilde{J}(\cdot) \in C[t_0, \infty)$.

Now we are able to define the operator

$$B(W, J, u_{C_0}) = (B_W(W, J, u_{C_0}), B_J(W, J, u_{C_0}), B_0(W, J, u_{C_0})),$$

$$M_W \times M_J \times M_0 \rightarrow M_W \times M_J \times M_0$$

by the formulas

$$B_W(W, J, u_{C_0}) := \begin{cases} J_0(t), & t \in [0, T]; \\ \frac{2e^{Rt/L} E(t) \sqrt{L}}{Z_0 + R_0} + \frac{Z_0 - R_0}{Z_0 + R_0} \tilde{J}(t), & t \in [T, \infty) \end{cases}$$

$$B_J(W, J, u_{C_0}) := \begin{cases} W_0(t), & t \in [0, T] \\ -\frac{2e^{Rt/L} u_{C_0}(t) \sqrt{L}}{Z_0 + R_2} + \frac{Z_0 - R_2}{Z_0 + R_2} \tilde{W}(t), & t \in [T, \infty) \end{cases},$$

$$B_0(W, J, u_{C_0}) := \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds - \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds$$

$$t \in [t_k, t_{k+1}], \quad (k = 0, 1, 2, \dots),$$

where

$$U_k(W, J, u_{C_0})(t) = (1 - (u_{C_0}(t) / \Phi))^{\gamma} / (C_{j_0})$$

$$\cdot \left[-I_{sat} \left(e^{au_{C_0}(t)} - 1 \right) + e^{-R(t-t_k)/L} (\tilde{W}(t) + J(t)) / (2\sqrt{L}) \right].$$

V. PRELIMINARY ASSERTIONS

Lemma 2. If $2 - |au_{C_0}(t)| \geq 2 - \alpha \phi_0 e^{\mu_0} > 0$, then (9) has a solution $(W, J, u_{C_0}) \in M_W \times M_J \times M_0$ iff the operator B has a fixed point in $M_W \times M_J \times M_0$, that is,

$$(W, J, u_{C_0}) = (B_0(B_W(W, J, u_{C_0}), B_J(W, J, u_{C_0}), B_0(W, J, u_{C_0}))).$$

(10)

Proof: Let $(V, I, u_{C_0}) \in M_V \times M_I \times M_0$ be a solution of (9). For the first and second equation the assertion is obvious. Then integrating the third equation of (9) on the interval $[t_k, t] \subset [t_k, t_{k+1}]$ ($k = 0, 1, 2, \dots$) we obtain

$$u_{C_0}(t) - u_{C_0}(t_k) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds$$

$$\Leftrightarrow u_{C_0}(t) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds.$$

To reach

$$u_{C_0}(t) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds - \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds$$

we notice

$$u_{C_0}(t) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds \Rightarrow 0 = u_{C_0}(t_{k+1}) = \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds.$$

Therefore the ordered triple (W, J, u_{C_0}) satisfies the system

$$W(t) = \left(2e^{Rt/L} E(t) \sqrt{L} + (Z_0 - R_0) J(t-T) \right) / (Z_0 + R_0),$$

$$J(t) = \left(-2e^{Rt/L} u_{C_0}(t) \sqrt{L} + (Z_0 - R_2) W(t-T) \right) / (Z_0 + R_0),$$

$$u_{C_0}(t) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds - \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds$$

for $t \in [t_k, t_{k+1}]$, that is, (W, J, u_{C_0}) is a fixed point of B .

Conversely, let (W, J, u_{C_0}) be a fixed point of B , that is, (W, J, u_{C_0}) satisfies (10). Then $(\mu_0 = \mu T_0 = \text{const})$ assuming $\mu > R/L$ and introducing $E_{\mu_0} = e^{(\mu - (R/L))T_0}$ we obtain

$$\begin{aligned} & \left| \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds \right| \leq \left(1/C_{j_0} \right) \left| \int_{t_k}^{t_{k+1}} \left(1 + |u_{C_0}(s)|/\Phi \right)^\gamma \right. \\ & \cdot \left[I_{sat} \left(e^{\alpha |u_{C_0}(s)|} - 1 \right) + (W(t-T) + |J(t)|) e^{-R(t-t_k)/L} / (2\sqrt{L}) \right] ds \Big| \\ & \leq \left(1/C_{j_0} \right) \left| \int_{t_k}^{t_{k+1}} \left(1 + \phi_0 e^{(\mu - (R/L))(t-t_k)} / \Phi \right)^\gamma \right. \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 e^{(\mu - (R/L))(t-t_k)}} - 1 \right) + (W_0 e^{-\mu T} + J_0) e^{(\mu - (R/L))(t-t_k)} / (2\sqrt{L}) \right] dt \Big| \\ & \leq \left(1 + \phi_0 E_{\mu_0} / \Phi \right)^\gamma / C_{j_0} \\ & \cdot \left| \int_{t_k}^{t_{k+1}} \left[I_{sat} \left(e^{\alpha \phi_0 e^{(\mu - (R/L))(t-t_k)}} - 1 \right) + (W_0 e^{-\mu T} + J_0) / (2\sqrt{L}) \right] e^{(\mu - (R/L))(t-t_k)} dt \right| \\ & \leq \left(1 + \phi_0 E_{\mu_0} / \Phi \right)^\gamma \left| \int_{t_k}^{t_{k+1}} e^{(\mu - (R/L))(t-t_k)} dt \right| / C_{j_0} \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 e^{(\mu - (R/L))(t_{k+1}-t_k)}} - 1 \right) + (W_0 e^{-\mu T} + J_0) / (2\sqrt{L}) \right] \\ & \leq \left(1 + \phi_0 E_{\mu_0} / \Phi \right)^\gamma \left(e^{(\mu - (R/L))(t_{k+1}-t_k)} - 1 \right) / C_{j_0} (\mu - (R/L)) \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 e^{-\mu T} + J_0) / (2\sqrt{L}) \right] \\ & \leq \left(1 + (\phi_0 E_{\mu_0} / \Phi)^\gamma \right) (E_{\mu_0} - 1) / C_{j_0} (\mu - (R/L)) \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 e^{-\mu T} + J_0) / (2\sqrt{L}) \right]. \end{aligned}$$

It follows $\int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds = 0$ because $\mu > 0$ could be chosen sufficiently large. Therefore

$$u_{C_0}(t) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds - \frac{t - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds$$

$$\Leftrightarrow u_{C_0}(t) = \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds.$$

Differentiating the last integral equation we obtain the third equation of (9).

Lemma 2 is thus proved.

VI. EXISTENCE-UNIQUENESS OF AN OSCILLATORY SOLUTION

Theorem 1. Let the following conditions be fulfilled:

- 1) $E(t) : [0, \infty) \rightarrow (-\infty, \infty)$, $E(t_k) = 0$, $|E(t)| \leq W_0 e^{(\mu - (R/L))(t-t_k)}$, $t \in [t_k, t_{k+1}]$, $(k = 0, 1, 2, \dots)$;
- 2) The initial functions $W_0(\cdot), J_0(\cdot) \in C[0, T]$ satisfy the conditions $W_0(\tau_k) = 0, J_0(\tau_k) = 0$ and $|W_0(t)| \leq W_0 e^{\mu(t-\tau_k)}$, $|J_0(t)| \leq J_0 e^{\mu(t-\tau_k)}$, $t \in [\tau_k, \tau_{k+1}]$, $(k = 0, 1, 2, \dots, m-1)$;
- 3) $2 - \alpha \phi_0 e^{\mu T_0} > 0$, $2\phi_0 \sqrt{C} \leq W_0$;
- 4) $(2\sqrt{L}W_0 + |Z_0 - R_0|)J_0 / (Z_0 + R_0) \leq W_0$;

- 5) $(Z_0 + |Z_0 - R_2|)W_0 / (Z_0 + R_2) \leq J_0$;
- 6) $(1 + (\phi_0 E_{\mu_0} / \Phi)^\gamma) E_{\mu_0} / C_{j_0} (\mu - (R/L)) \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0) / (2\sqrt{L}) \right] \leq \phi_0$;
- 7) $K_J = (2\sqrt{L} + |Z_0 - R_2|) / (Z_0 + R_2) < 1$;
- 8) $K_C = E_{\mu_0} (1 + (\phi_0 E_{\mu_0} / \Phi)^\gamma) / (\mu C_{j_0}) \cdot \left\{ \left(\gamma / \Phi \right) \left(I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (W_0 + J_0) / (2\sqrt{L}) \right) \right\} \cdot \left(1 - (\phi_0 E_{\mu_0} / \Phi) \right)^{-1} (E_{\mu_0} - 1) / 2 + I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + 1 / \sqrt{L} \Big\} < 1$.

Then there exists a unique oscillatory solution of (9).

Proof: First we show that

$$B = (B_W(W, J, u_{C_0}), B_J(W, J, u_{C_0}), B_0(W, J, u_{C_0}));$$

$$M_W \times M_J \times M_0 \rightarrow M_W \times M_J \times M_0.$$

Indeed, it is easy to see that $B_W(W, J, u_{C_0})(t)$ and

$B_J(W, J, u_{C_0})(t)$ are oscillatory functions. Indeed,

$$\begin{aligned} & B_0^{(k-1)}(W, J, u_{C_0})(t_k) \\ & = \int_{t_{k-1}}^{t_k} U_{k-1}(W, J, u_{C_0})(s) ds - \frac{t_k - t_{k-1}}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} U_{k-1}(W, J, u_{C_0})(s) ds = 0, \\ & B_0^{(k)}(W, J, u_{C_0})(t_k) \\ & = \int_{t_k}^{t_k} U_k(W, J, u_{C_0})(s) ds - \frac{t_k - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds = 0. \end{aligned}$$

To verify their continuity we notice

$$\begin{aligned} W_0(T) &= \frac{2e^{RT/L} E(T) \sqrt{L}}{Z_0 + R_0} + \frac{Z_0 - R_0}{Z_0 + R_0} J(0) \\ J_0(T) &= -\frac{2e^{RT/L} u_{C_0}(T) \sqrt{L}}{Z_0 + R_2} + \frac{Z_0 - R_2}{Z_0 + R_2} W(0), \end{aligned}$$

$$\begin{aligned} & B_0^{(k)}(W, J, u_{C_0})(t_{k+1}) = \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds \\ & - \frac{t_{k+1} - t_k}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds = 0, \\ & B_0^{(k+1)}(W, J, u_{C_0})(t_{k+1}) = \int_{t_{k+1}}^{t_{k+1}} U_{k+1}(W, J, u_{C_0})(s) ds \\ & - \frac{t_{k+1} - t_{k+1}}{t_{k+2} - t_{k+1}} \int_{t_{k+1}}^{t_{k+2}} U_{k+1}(W, J, u_{C_0})(s) ds = 0, \\ & t \in [t_k, t_{k+1}], (k = 0, 1, 2, \dots). \end{aligned}$$

The following inequalities are valid for $t \in [t_k, t_{k+1}]$,

$(k = 0, 1, 2, \dots)$:

$$\begin{aligned} & |B_W(W, J, u_{C_0})(t)| \\ & \leq (2e^{Rt/L} |E(t)| \sqrt{L} + |Z_0 - R_0| |J(t-T)|) / (Z_0 + R_0) \\ & \leq (2W_0 e^{\mu(t-t_k)} \sqrt{L} + |Z_0 - R_0| J_0 e^{\mu(t-t_k)} e^{-\mu T}) / (Z_0 + R_0) \\ & \leq (2W_0 \sqrt{L} + |Z_0 - R_0| J_0) e^{\mu(t-t_k)} / (Z_0 + R_0) \leq W_0 e^{\mu(t-t_k)} \\ & \text{and} \\ & |B_J(W, J, u_{C_0})(t)| \\ & \leq (2e^{Rt/L} u_{C_0}(t) \sqrt{L} + |Z_0 - R_2| |W(t-T)|) / (Z_0 + R_2) \\ & \leq (2\sqrt{L} \phi_0 + |Z_0 - R_2| W_0 e^{-\mu T}) e^{\mu(t-t_k)} / (Z_0 + R_2) \end{aligned}$$

$$\leq (2Z_0\sqrt{C}\phi_0 + |Z_0 - R_2|W_0e^{-\mu T})e^{\mu(t-t_k)}/(Z_0 + R_2) \\ \leq e^{\mu(t-t_k)}(Z_0 + |Z_0 - R_2|W_0)/(Z_0 + R_2) \leq J_0e^{\mu(t-t_k)}.$$

Using (10) we obtain

$$\begin{aligned} & |B_0(W, J, u_{C_0})(t)| \leq \left| \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds \right| + \\ & + |(t - t_k)/(t_{k+1} - t_k)| \left| \int_{t_k}^{t_{k+1}} U_k(W, J, u_{C_0})(s) ds \right|; \\ & \left| \int_{t_k}^t U_k(W, J, u_{C_0})(s) ds \right| \leq (1/C_{j0}) \left| \int_{t_k}^t \left(1 + (|u_{C_0}(s)|/\Phi)^\gamma \right. \right. \\ & \cdot \left. \left[I_{sat} \left(e^{\alpha |u_{C_0}(s)|} - 1 \right) + (|W(s-T)| + |J(s)|) e^{-R(s-t_k)/L} / (2\sqrt{L}) \right] ds \right| \\ & \leq (1/C_{j0}) \left| \int_{t_k}^t \left(1 + \phi_0 e^{(\mu-(R/L))(s-t_k)}/\Phi \right)^\gamma \right. \\ & \cdot \left. \left[I_{sat} \left(e^{\alpha \phi_0 e^{(\mu-(R/L))(s-t_k)}} - 1 \right) + (W_0 + J_0) e^{(\mu-(R/L))(s-t_k)}/(2\sqrt{L}) \right] ds \right| \\ & \leq \left(1 + \phi_0 e^{(\mu-(R/L))T_0}/\Phi \right)^\gamma / C_{j0} \\ & \cdot \left| \int_{t_k}^t \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0)/(2\sqrt{L}) \right] e^{(\mu-(R/L))(s-t_k)} ds \right| \\ & \leq \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0)/(2\sqrt{L}) \right] / C_{j0} \\ & \cdot \left(1 + \phi_0 E_{\mu_0}/\Phi \right)^\gamma \left| \int_{t_k}^t e^{(\mu-(R/L))(s-t_k)} ds \right| \\ & \leq \left(1 + \phi_0 E_{\mu_0}/\Phi \right)^\gamma \left(e^{(\mu-(R/L))(t-t_k)} - 1 \right) / C_{j0} (\mu - (R/L)) \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0)/(2\sqrt{L}) \right]. \end{aligned}$$

In a similar way

$$\begin{aligned} & \left| \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} U(W, J, u_{C_0})(s) ds \right| \\ & \leq \left(1 + \phi_0 E_{\mu_0}/\Phi \right)^\gamma (E_{\mu_0} - 1) / (\mu - (R/L)) \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0)/(2\sqrt{L}) \right] / C_{j0}. \end{aligned}$$

Therefore

$$\begin{aligned} & |B_0(W, J, u_{C_0})(t)| \leq \\ & \leq (1/C_{j0}) \left(1 + \phi_0 E_{\mu_0}/\Phi \right)^\gamma \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0)/(2\sqrt{L}) \right] \\ & \cdot \left(e^{(\mu-(R/L))(t-t_k)} + (E_{\mu_0} - 1) e^{(\mu-(R/L))(t-t_k)} \right) / (\mu - (R/L)) \\ & \leq (1/C_{j0}) \left(1 + \phi_0 E_{\mu_0}/\Phi \right)^\gamma \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0)/(2\sqrt{L}) \right] \\ & \cdot E_{\mu_0} e^{(\mu-(R/L))(t-t_k)}/(\mu - (R/L)) \leq \phi_0 e^{(\mu-(R/L))(t-t_k)}. \end{aligned}$$

Therefore B maps $M_W \times M_J \times M_0$ into itself.

It remains to show that B is contractive operator. Indeed, for $t \in [t_k, t_{k+1}]$, $(k = 0, 1, 2, \dots)$ we obtain

$$\begin{aligned} & |B_W(W, J, u_{C_0})(t) - B_W(\bar{W}, \bar{J}, \bar{u}_{C_0})(t)| \\ & \leq |Z_0 - R_0| |J(t-T) - \bar{J}(t-T)| / (Z_0 + R_0) \\ & \leq |Z_0 - R_0| |J(t-T) - \bar{J}(t-T)| e^{-\mu(t-T-t_k)} e^{\mu(t-T-t_k)} / (Z_0 + R_0) \\ & \leq |Z_0 - R_0| \rho_s(J, \bar{J}) e^{\mu(t-t_k)} / (Z_0 + R_0). \end{aligned}$$

It follows

$$\rho_k(B_W(W, J, u_{C_0}), B_W(\bar{W}, \bar{J}, \bar{u}_{C_0})) \leq |Z_0 - R_0| \rho_s(J, \bar{J}) / (Z_0 + R_0).$$

In a similar way

$$\begin{aligned} & |B_J(W, J, u_{C_0})(t) - B_J(\bar{W}, \bar{J}, \bar{u}_{C_0})(t)| \\ & \leq (2\sqrt{L} |u_{C_0}(t) - \bar{u}_{C_0}(t)| + |Z_0 - R_2| |W(t-T) - \bar{W}(t-T)|) / (Z_0 + R_2) \\ & \leq (2\sqrt{L} \rho_k(u_{C_0}, \bar{u}_{C_0}) + |Z_0 - R_2| \rho_s(W, \bar{W})) e^{\mu(t-t_k)} / (Z_0 + R_2). \end{aligned}$$

It follows

$$\begin{aligned} & \rho_k(B_J(W, J, u_{C_0}), B_J(\bar{W}, \bar{J}, \bar{u}_{C_0})) \\ & \leq (2\sqrt{L} \rho_k(u_{C_0}, \bar{u}_{C_0}) + |Z_0 - R_2| \rho_s(W, \bar{W})) / (Z_0 + R_2) \\ & \leq (2\sqrt{L} + |Z_0 - R_2|) \max\{\rho_k(u_{C_0}, \bar{u}_{C_0}), \rho_s(W, \bar{W})\} / (Z_0 + R_2). \end{aligned}$$

Finally for $0 < \gamma < 1$ we have

$$\begin{aligned} & |B_0^{(k)}(V, I, u_{C_0})(t) - B_0^{(k)}(\bar{V}, \bar{I}, \bar{u}_{C_0})(t)| \\ & \leq \int_{t_k}^t |U_k(V, I, u_{C_0})(s) - U_k(\bar{V}, \bar{I}, \bar{u}_{C_0})(s)| ds \\ & + \left| \frac{t-t_k}{t_{k+1}-t_k} \int_{t_k}^{t_{k+1}} U_k(V, I, u_{C_0})(s) ds - \int_{t_k}^{t_{k+1}} U_k(\bar{V}, \bar{I}, \bar{u}_{C_0})(s) ds \right| \\ & \equiv W_1 + W_2. \end{aligned}$$

Since

$$\begin{aligned} W_1 & \leq \int_{t_k}^t \left| \left(1 - (u_{C_0}(s)/\Phi)^\gamma \right) - \left(1 - (\bar{u}_{C_0}(s)/\Phi)^\gamma \right) e^{(\mu-(R/L))(s-t_k)} \right. \\ & \cdot \left. \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 e^{-\mu T} + J_0)/(2\sqrt{L}) \right] ds \right. \\ & + (1/C_{j0}) \left(1 + (\phi_0 E_{\mu_0}/\Phi)^\gamma \int_{t_k}^t e^{(\mu-(R/L))(s-t_k)} ds \right. \\ & \cdot \left(\alpha I_{sat} e^{\alpha \phi_0 E_{\mu_0}} \rho_k(u_{C_0}, \bar{u}_{C_0}) + (\rho_s(W, \bar{W}) + \rho_k(J, \bar{J})) / (2\sqrt{L}) \right) \\ & \leq (\gamma/C_{j0}) \left(1 - (\phi_0 E_{\mu_0}/\Phi)^\gamma \right)^{-1} \\ & \cdot \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} - 1) + (W_0 + J_0)/(2\sqrt{L}) \right) \\ & \cdot \int_{t_k}^t |u_{C_0}(s) - \bar{u}_{C_0}(s)| / \Phi e^{(\mu-(R/L))(s-t_k)} ds \\ & + (1/C_{j0}) \left(1 + (\phi_0 E_{\mu_0}/\Phi)^\gamma \int_{t_k}^t e^{(\mu-(R/L))(s-t_k)} ds \right. \\ & \cdot \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} \rho_k(u_{C_0}, \bar{u}_{C_0}) + (\rho_s(W, \bar{W}) + \rho_k(J, \bar{J})) / (2\sqrt{L})) \right) \\ & \leq \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} - 1) + (W_0 e^{-\mu T} + J_0)/(2\sqrt{L}) \right) \\ & \cdot (\gamma/C_{j0}) \left(1 - (\phi_0 E_{\mu_0}/\Phi)^\gamma \right)^{-1} \rho_k(u_{C_0}, \bar{u}_{C_0}) \int_{t_k}^t e^{2(\mu-(R/L))(s-t_k)} ds \\ & + (1/C_{j0}) \left(1 + (\phi_0 E_{\mu_0}/\Phi)^\gamma \int_{t_k}^t e^{(\mu-(R/L))(s-t_k)} ds \right. \\ & \cdot \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} \rho_k(u_{C_0}, \bar{u}_{C_0}) + (\rho_s(W, \bar{W}) + \rho_k(J, \bar{J})) / (2\sqrt{L})) \right) \\ & \leq (\gamma/C_{j0}\Phi) \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} - 1) + (W_0 e^{-\mu T} + J_0)/(2\sqrt{L}) \right) \end{aligned}$$

$$\cdot \left(1 - (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma-1} \rho_k(u_{C_0}, \bar{u}_{C_0}) \left(e^{2(\mu-(R/L))(t-t_k)} - 1\right) / (2\mu) \\ + (1/(C_{j_0})) \left(1 + (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma} \left(e^{(\mu-(R/L))(t-t_k)} - 1\right) / \mu \\ \cdot \left(I_{sat} e^{\alpha \phi_0 E_{\mu_0}} \rho_k(u_{C_0}, \bar{u}_{C_0}) + (\rho_s(W, \bar{W}) + \rho_k(J, \bar{J})) / (2\sqrt{L})\right) \\ \leq 1/(\mu C_{j_0}) e^{(\mu-(R/L))(t-t_k)}$$

$$\cdot \left\{ \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} - 1) + (W_0 + J_0) / (2\sqrt{L}) \right) \right. \\ \cdot (\gamma / \Phi) \left(1 - (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma-1} (E_{\mu_0} - 1) / 2 \\ \left. + \left(1 + (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma} \left[I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (1/\sqrt{L}) \right] \right\} \\ \cdot \max \{ \rho_k(u_{C_0}, \bar{u}_{C_0}), \rho_s(W, \bar{W}), \rho_k(J, \bar{J}) \}$$

and

$$W_2 \leq (1/(\mu C_{j_0})) (E_{\mu_0} - 1) \\ \cdot \left\{ \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} - 1) + (W_0 + J_0) / (2\sqrt{L}) \right) (\gamma / \Phi) \right. \\ \cdot e^{(\mu-(R/L))(t-t_k)} \left(1 - (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma-1} (E_{\mu_0} - 1) / 2 \\ \left. + \left(1 + (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma} \left[I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (1/\sqrt{L}) \right] \right\} \\ \cdot \max \{ \rho_k(u_{C_0}, \bar{u}_{C_0}), \rho_s(W, \bar{W}), \rho_k(J, \bar{J}) \}$$

then

$$\left| B_0^{(k)}(W, J, u_{C_0})(t) - B_0^{(k)}(\bar{W}, \bar{J}, \bar{u}_{C_0})(t) \right| \\ \leq e^{(\mu-(R/L))(t-t_k)} E_{\mu_0} / (\mu C_{j_0}) \\ \cdot \left\{ \left(I_{sat} (e^{\alpha \phi_0 E_{\mu_0}} - 1) + (W_0 + J_0) / (2\sqrt{L}) \right) (\gamma / \Phi) \right. \\ \cdot \left(1 - (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma-1} (E_{\mu_0} - 1) / 2 \\ \left. + \left(1 + (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma} \left[I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (1/\sqrt{L}) \right] \right\} \\ \cdot \max \{ \rho_k(u_{C_0}, \bar{u}_{C_0}), \rho_s(W, \bar{W}), \rho_k(J, \bar{J}) \}.$$

Therefore

$$\rho_k(B_0^{(k)}(W, J, u_{C_0}), B_0^{(k)}(\bar{W}, \bar{J}, \bar{u}_{C_0})) \\ \leq K_C \rho_{j(k)}((W, J, u_{C_0}), (\bar{W}, \bar{J}, \bar{u}_{C_0}))$$

where

$$K_C = E_{\mu_0} \left(1 + \phi_0 E_{\mu_0} / \Phi\right)^{\gamma} / (\mu C_{j_0}) (\gamma / \Phi) \\ \cdot \left\{ \left(I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (W_0 + J_0) / (2\sqrt{L}) \right) \left(1 - \phi_0 E_{\mu_0} / \Phi\right)^{\gamma-1} (E_{\mu_0} - 1) / 2 \right. \\ \left. + I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (1/\sqrt{L}) \right\}.$$

Finally we obtain

$$\rho_k((B_W(W, J, u_{C_0}), B_J(W, J, u_{C_0}), B_0(W, J, u_{C_0})), \\ (B_W(\bar{W}, \bar{J}, \bar{u}_{C_0}), B_J(\bar{W}, \bar{J}, \bar{u}_{C_0}), B_0(\bar{W}, \bar{J}, \bar{u}_{C_0}))) \\ \leq L_0 \rho_{j(k)}((W, J, u_{C_0}), (\bar{W}, \bar{J}, \bar{u}_{C_0}))$$

where $L_0 = \max\{K_W, K_J, K_C\} < 1$.

Therefore the fixed point of operator B is an oscillatory solution of the problem (9).

Theorem 1 is thus proved.

VII. NUMERICAL EXAMPLE

Here we rewrite all inequalities from the main theorem guaranteeing existence-uniqueness of oscillatory solution:

$$\begin{aligned} & (2\sqrt{C} Z_0 W_0 + |Z_0 - R_0| J_0) / (Z_0 + R_0) \leq W_0; \\ & (2\sqrt{L} \phi_0 + |Z_0 - R_2| W_0) / (Z_0 + R_2) \leq J_0; \\ & (1/C_{j_0}) \left(1 + \phi_0 E_{\mu_0} / \Phi\right)^{\gamma} \\ & \cdot \left[I_{sat} \left(e^{\alpha \phi_0 E_{\mu_0}} - 1 \right) + (W_0 + J_0) / (2\sqrt{L}) \right] \\ & \cdot E_{\mu_0} / (\mu - (R/L)) \leq \phi_0; \\ & K_W = |Z_0 - R_0| / (Z_0 + R_0) < 1; \\ & K_J = (2\sqrt{L} + Z_0 - R_2) / (Z_0 + R_2) < 1; \\ & K_C = E_{\mu_0} / (\mu C_{j_0}) \\ & \cdot \left\{ \left(I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (W_0 + J_0) / (2\sqrt{L}) \right) (\gamma / \Phi) \right. \\ & \cdot \left(1 - (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma-1} (E_{\mu_0} - 1) / 2 \\ & \left. + \left(1 + (\phi_0 E_{\mu_0} / \Phi)\right)^{\gamma} \left[I_{sat} e^{\alpha \phi_0 E_{\mu_0}} + (1/\sqrt{L}) \right] \right\} < 1. \end{aligned}$$

Let us consider a transmission line with length $\Lambda = 10$ m.

The inductance, capacitance and resistance per-unit length are $L = 0,45 \mu\text{H}/\text{m}$; $C = 80 \text{ pF}/\text{m}$; $R = 0,01 \Omega/\text{m}$; $R/L = 2,2 \cdot 10^4$; $R_0 = R_2 = 35 \Omega$. Then in order to satisfy the Heaviside condition we have to choose

$$G = LC/R = 0,36 \cdot 10^{-12} \text{ mho}/\text{m}; T = \Lambda \sqrt{LC} = 6 \cdot 10^{-8};$$

$$Z_0 = \sqrt{L/C} = 75 \Omega. \text{ Here } \alpha = q/(\eta KT), \text{ where}$$

$q = 1,6 \cdot 10^{-19} \text{ C}$ is the electron charge, $K = 1,23 \cdot 10^{-23}$ is the Boltzmann constant; $T = 293 \text{ K}$ (20°C) is absolute temperature. We choose the ideality factor $\eta \in [1; 1,2]$ (cf. [2]).

The diffusion potential is $\Phi = 0,5 \text{ V}$, $C_{j_0} = 5 \cdot 10^{-11} \text{ F}$ is the zero-voltage junction capacitance, $I_{sat} = 10^{-8} \text{ A}$,

$\alpha = 37$. If the junction is uniformly doped, $\gamma = 0,5$. For $T_0 = 10^{-16}$ we choose $\mu = 10^{14} \Rightarrow (\mu - (R/L))T_0 \approx 10^{-2}$,

$$E_{\mu_0} = e^{(\mu-(R/L))T_0} = e^{0,01} \approx 1 \text{ and we take } \phi_0 = 0,16. \text{ Then}$$

$$\alpha \phi_0 = 37 \cdot 0,16 = 5,92 \Rightarrow e^{5,92} \approx 372,5 \text{ and we choose}$$

$$W_0 = J_0 = 10^{-4}. \text{ The above inequalities become}$$

$$2\sqrt{80 \cdot 10^{-12}} \frac{75}{75+15} = 2 \cdot 10^{-6} \sqrt{80} \frac{75}{75+15} \leq 1;$$

$$2\sqrt{0,45 \cdot 10^{-6}} \frac{0,16}{75+15} \leq 10^{-4};$$

$$\frac{0,5(1,32)^{1,5}}{10^{14} \cdot 1,5 \cdot 5 \cdot 10^{-11}} \left(10^{-8}(373) + \frac{10^{-4}}{2\sqrt{0,45 \cdot 10^{-6}}} \right) \leq 0,16;$$

$$K_W = \frac{75-35}{75+35} = 0,36 < 1; \quad K_J = 0,36 < 1; \quad L_0 = 0,36;$$

$$K_C = (1/\mu C_{j_0}) \left(1 + (\phi_0 / \Phi)\right)^{\gamma} \left(I_{sat} \exp(\alpha \phi_0) + (1/\sqrt{L}) \right) \approx 0,34.$$

VIII. CONCLUSION

We have noticed that the transmission lines loaded by Schottky diode circuit requires a specific way of derivation of the boundary conditions and respectively leads to different type of differential equations, namely, one could not exclude some transitional currents and so we have to solve a system of three equations for three unknown functions. We have shown that the right-hand side of the differential equations are not periodic functions that implies to look for oscillatory solutions. Introducing a suitable operator whose fixed point are oscillatory solutions we obtain existence-uniqueness of an oscillatory solution. It can be reached by successive approximations as in [8] from the operator equations

$$W^{(p)}(t) = \left(2e^{Rt/L} E(t) \cdot 10^{-3} \sqrt{0,45} + 40J^{(p-1)}(t-T) \right) / 110,$$

$$J^{(p)}(t) =$$

$$= \left(-2e^{Rt/L} u_{C_0}^{(p-1)}(t) \cdot 10^{-3} \sqrt{0,45} + 40W^{(p-1)}(t-T) \right) / 110$$

$$u_{C_0}^{(p)}(t) = B_0(W^{(p-1)}, J^{(p-1)}, u_{C_0}^{(p-1)})$$

beginning with suitable $W^{(0)}(t), J^{(0)}(t), u_{C_0}^{(0)}$. The rate of

convergence is

$$\begin{aligned} & \rho_k \left(\left(W^{(p+1)}, J^{(p+1)}, u_{C_0}^{(p+1)} \right), \left(W^{(p)}, J^{(p)}, u_{C_0}^{(p)} \right) \right) \leq \\ & \leq \frac{(0,36)^p}{1-0,36} \rho_k \left(\left(W^{(1)}, J^{(1)}, u_{C_0}^{(1)} \right), \left(W^{(0)}, J^{(0)}, u_{C_0}^{(0)} \right) \right). \end{aligned}$$

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