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REVIEW

Electric Field Derivation from the Scalar Potential of a Filamentary Plane Uniformly Charged Ring at Any Point in Space

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ABSTRACT

The growing popularity of technological devices has led to renewed interest in the electric fields generated by various configurations of charged elements. Early physicists calculated these fields, but only for the simplest cases. Several textbooks on electrostatics have calculated the off-axis positions of rings, and these solutions can be obtained using approximate expressions. The major difficulty in calculating the field for nearly all configurations is that the equation cannot be solved without using special functions, such as elliptic integrals. Althoug many of these are tabulated, the calculations are laborious. However, such calculations can be performed using computers, because software programs or subroutines can be written for many of these special functions. This study investigates the derivation of the electric field from the electric potential created by filamentary plane uniformly charged rings at arbitrary points in space. The field expressions were obtained as functions of the complete elliptic integrals of the first and second kinds. Our method provides a basis for studying the advanced electrostatics of uniformly charged rings.

Keywords: Electric field vector, electric potential, elliptic integrals, uniformly charged ring.

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1. Introduction

Most modern textbooks on Electrostatics begin by introducing Coulomb's law, electrostatic field, and scalar potential of discrete charge distributions because they are comparatively easy to understand, and only later do they consider continuous charge distributions. These methods often involve complex and advanced calculations using specific mathematical functions. So that students can assimilate the calculations of these physical quantities, teaching methods are focused on case studies of symmetric continuous charge distributions having a center, axes, and planes of symmetry, such as uniformly charged straight lines, rings, annulus, disks, planes, cylinders, and spheres, and cylindrical and spherical shells. All it takes is a book on Electrostatics. These common examples are available on several websites. Therefore, calculating the electric field vector created at any point in space using such continuous distributions is relatively simple. It is sufficient to apply Coulomb's law or Gauss's theorem to these distributions to find the expression of the electric field created everywhere in space. However, the problem is not the same when calculating the electric field vector or scalar potential of a uniformly charged ring or disk at any point in space. In these cases, the calculation is complex. Calculations for thin charged rings or disks are straightforward only when performed at points generally laying along their axis [1, p. 14], [2, p. 707], [3, p. 697] or any other textbooks on the Electrostatics or Electromagnetism.

A brief review of the existing literature on the off-axis electric field of a uniform ring of charge, the first study on the off-axis field of a uniformly charged ring was reported in 2005 by Zypman [4]. In particular, the author considered the problem of finding the electrostatic field produced by a ring of charge at any point outside its axis. In 2007, Mandre published a calculation of the off-axis electric field of a uniform ring of charge using special functions such as complete integrals [5]. The author introduced expressions for the radial and axial components of the electric field using overly concentrated and laborious calculations. In 2009, Ciftja et al. calculate the electrostatic potential of a single uniformly charged ring at an arbitrary off-axis point using two methods [6]. The "direct integration" is the most suitable of them. Subsequently, Wissner-Gross published a video on YouTube that explained how to calculate the electric field in the plane of a uniformly charged ring using a simplified approach based on Euclid's elements [7]. This approach was designed for undergraduates. Publishing his contribution, Escalante derived the field from the electrostatic potential along the axis of a non-centered circular uniformly charged ring [8]. The author explained how to obtain the potential and deduce the electric field from it through differentiation. Recently, calculations of the electric fields and forces for measuring the electric charge of a ring using canning force microscopy [9]–[11] have been reported, and this is an example of the application of a charged ring, along with numerous others. The issue of studying the same physical quantities of electrostatics for charged disks [12]–[14] and annuli [15], [16] is not as obvious as that for rings. In these cases, the calculations are laborious.

The aim in reviewing published articles on this topic was to develop a unified theory using our original approach. In this study, we demonstrate how to derive the electric field vector from the gradient of the scalar potential of the ring at any point in space. First, we establish the expression of the potential scalar function at any point in space using Coulomb's law. Next, we derive the electric field by applying the gradient operator to the potential of the ring. This is a standard and robust method in Electrostatics. Moreover, this approach involves the complete elliptic integrals of the first, second and third kinds. The study represents a conceptual advance by providing the most compact possible expressions for the radial and axial components of the electric field of a uniformly charged ring using complete elliptic integrals. While previous works have touched upon this, the paper's explicit derivation and presentation of these compact forms, particularly linking them to the third kind elliptic integral, offer a valuable consolidation.

The pedagogical approach we present is the safest way to solve the problem of determining the electric field vector from the scalar electric potential of a thin, planar, and uniformly charged ring.

2. Materials and Methods

2.1. Electric Potential Calculation at Any Point in Space

A thin ring of radius R carries a total charge Q, which is algebraic and uniformly distributed around it.

The linear charge density is $\lambda = Q/(2\pi R)$. A position vector $\boldsymbol{r}$ points to a point P that lies in the 3D-space, as shown in Fig. 1.

The scalar electric potential $dV(r)$ at point P in space due to a small single segment of charge dq of an arc of the ring is given by Coulomb's law:

$$dV(r) = k_e \frac{dq}{\|\boldsymbol{r} - \boldsymbol{r}'\|} \tag{1}$$

where $k_e = 1/(4\pi\varepsilon_0)$ is the Coulomb constant, ε_0 is the vacuum permittivity, $\varepsilon_0 = 8.854 \times 10^{-12}$ F·m^{-1}.

The ($\boldsymbol{r} - \boldsymbol{r}'$) vector is the displacement vector pointing from point P′ to point P.

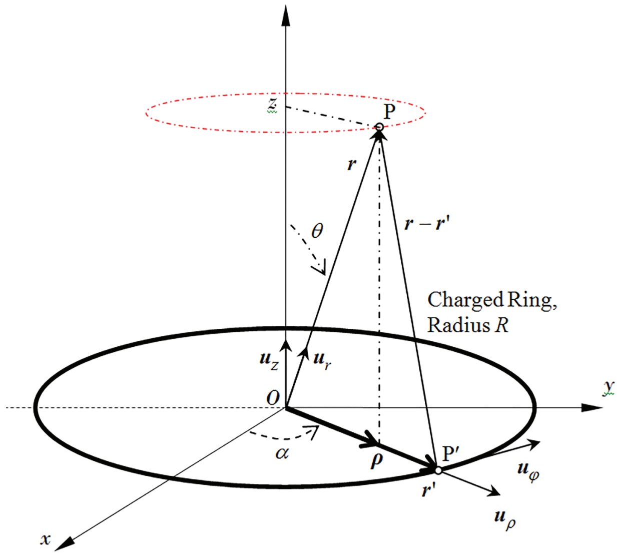

Fig. 1. The charged ring that lies on the xy plane produces an electric field vector at the point P.

The small charge element, dq, at point P′ is indicated by the position vector $\boldsymbol{r}'$ which forms an angle α with the x axis. The projections in the Cartesian coordinate system are as follows:

$$\boldsymbol{r}' = R\cos\alpha\, \boldsymbol{i} + R\sin\alpha\, \boldsymbol{j} \tag{2}$$

However, in cylindrical coordinates, the position vector $\boldsymbol{r}$ can be written from the spherical coordinates in the rz plane as:

$$\begin{aligned} \boldsymbol{r} &= r\boldsymbol{u}_r \\ &= \rho\, \boldsymbol{u}_\rho + z\, \boldsymbol{u}_z \end{aligned} \tag{3}$$

with

$$\cos(\theta) = \frac{\boldsymbol{r}.\boldsymbol{k}}{r},\ \ \rho = r\sin\theta,\ \ z = r\cos\theta \tag{4}$$

Invariances study shows that for fixed values of the (ρ, z) coordinates, the scalar potential $V(r)$ does not depend on the azimuthal angle ϕ when it varies. Consequently, $V(r)$ at any point P on any circle centered on the z axis is invariant to the angle ϕ. Thus, it is possible to calculate $V(r)$ at any point P in space independent of ϕ.

On can solve the problem by choosing any plane passing through point P and the z axis owing to symmetry. Therefore, the symmetry and invariance study of the ring system demonstrated that we do not lose generality by calculating the electric vector in the xz plane at any $\boldsymbol{r}$ (ρ, 0, z) position vector. $V(r)$ will be the same at any angle ϕ for the same pair of values of fixed (ρ, z) coordinates by symmetry. Thus, the calculation can be performed, for the time being in the xz plane, rather than any other plane. This is achieved simply by replacing $\boldsymbol{u}_\rho$ and $\boldsymbol{u}_z$ in position vector (3) with unit vectors $\boldsymbol{i}$ and $\boldsymbol{k}$ respectively, as follows:

$$\boldsymbol{r} = \rho\, \boldsymbol{i} + z\, \boldsymbol{k} \tag{5}$$

Thus, we consider component $\rho\, \boldsymbol{u}_\rho$ of position vector $\boldsymbol{r}$ to lie on the plane of the ring along the x axis.

Therefore, by using the position vectors defined in (2) and (5), the displacement vector from P′ to P is

$$\boldsymbol{r} - \boldsymbol{r}' = (\rho - R\cos\alpha)\,\boldsymbol{i} - R\sin\alpha\,\boldsymbol{j} + z\,\boldsymbol{k} \tag{6a}$$

Then, the full distance between P and P′ is

$$\left\|\boldsymbol{r} - \boldsymbol{r}'\right\| = \left(R^2 + \rho^2 + z^2 - 2R\rho\cos\alpha\right)^{1/2} \tag{6b}$$

Substituting expression (6b) in Coulomb's law, given by (1) results in

$$dV(r) = k_e \frac{dq}{\left(R^2 + \rho^2 + z^2 - 2R\rho\cos\alpha\right)^{1/2}} \tag{7}$$

Therfore, it is no longer necessary to restrict the calculation to the xz plane. Instead, any rz plane can be considered symmetric. In addition, becuase the charge Q is distributed uniformly over the circumference of the ring, that is, i.e., as $dq = \lambda d\ell = \lambda R d\alpha = Q/(2\pi R)\ R d\alpha = Q/(2\pi)d\alpha$, integration can be performed over the ring circumference. Given these considerations, the expression of the scalar function of the electric potential is

$$V(\rho, z) = k_e \frac{Q}{2\pi} \int_0^{2\pi} \frac{d\alpha}{\left(R^2 + \rho^2 + z^2 - 2R\rho\cos\alpha\right)^{1/2}} \tag{8}$$

This function adequately describes the behavior of the scalar potential of a uniformly charged ring, and is sufficient to calculate a series of values using a computer and stop. Function (8) can also be expressed in terms of the complete elliptic integral of the first kind. To do this, one can manually process the data as follows.

Because cos(α) is a periodic function with period 2π, function (8) can be calculated only as

$$V(\rho, z) = k_e \frac{Q}{\pi} \int_0^{\pi} \frac{d\alpha}{\left(R^2 + \rho^2 + z^2 - 2R\rho\cos\alpha\right)^{1/2}} \tag{9}$$

Integration by substitution is used, and the integration variable must be changed according to the following change in variables:

$$\beta = \frac{\pi - \alpha}{2}, \text{ such that } \alpha = \pi - 2\beta \text{ and } d\alpha = -2d\beta \tag{10}$$

$$\alpha = 0 \rightarrow \beta = \frac{\pi}{2},\ \alpha = \pi \rightarrow \beta = 0, \cos(\alpha) = \cos(\pi - 2\beta)$$
$$= -\cos(2\beta) = 2\sin^2(\beta) - 1 \tag{11}$$

By substituting, function (9) can be rewritten as:

$$V(\rho, z) = k_e Q \frac{2}{\pi} \int_0^{\pi/2} \frac{d\beta}{\left(\rho^2 + R^2 + z^2 + 2\rho R - 4\rho R sin^2\beta\right)^{1/2}} \tag{12}$$

Then, by introducing

$$\mu = k^2 = \frac{4\rho R}{q} \text{ where } q = R^2 + \rho^2 + z^2 + 2\rho R \tag{13}$$

where the quantity $k = \sqrt{\mu}$ is the modulus of Jacobian elliptic functions [17], and the physical restriction in (13) bounds k to the interval [0], [1] according to the textbooks on the special functions. Substituting, the scalar function of the electric potential yields the following expression.

$$V\left(q, k^2\right) = k_e \frac{2}{\pi} \frac{Q}{\sqrt{q}} \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 sin^2\beta}} \tag{14}$$

In this expression, we can recognize the form of the complete elliptic integral of the first kind, often denoted $K(k^2)$. This integral is defined as follows:

$$K\left(k^2\right) = \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 sin^2\beta}} \tag{15}$$

which is defined for $0 \le k \le 1$ as in Arfken and Weber [18]. At $k = 0$, $K(0) = \pi/2$; when $k \rightarrow 1$, $K(k^2) \rightarrow \infty$.

The complete elliptic integral of the first kind can be written as a power series [19]:

$$K\left(k^2\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}(n!)^2}\right) k^{2n}$$
$$= \frac{\pi}{2} \sum_{n=0}^{\infty} \left(P_{2n}(0)\right)^2 k^{2n}, \tag{16}$$

where P_n is the Legendre polynomials, which is equivalent to

$$K\left(k^2\right) = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \ldots + \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 k^{2n} + \ldots\right) \tag{17}$$

where $(n)!!$ denotes the double factorial.

Thus, reorganizing the function of the potential leads to the Legendre polynomial offering an alternative approach for the approximate calculation of the electric potential [20].

From (14), the most compact form of the electric potential is

$$V\left(q, k^2\right) = k_e \frac{2}{\pi} \frac{Q}{\sqrt{q}} K\left(k^2\right) \tag{18}$$

The complete elliptic integral of the first kind defined by function (15) is a continuous function that increase from $K(0) = \pi/2$ to $K(1) \rightarrow \infty$ [18], meaning that function (18) of the potential is therefore differentiable into this domain of definition.

2.2. *Electric Field Derivation from the Scalar Potential*

According to the symmetry study, the electric field vector of a uniformly charged ring should contain only two components: radial and axial, while the azimuthal component is zero. Moreover, the invariance study demonstrates that the electric field vector is independent of the azimuthal angle ϕ, and it only depends on the ρ and z cylindrical coordinates. Accordingly, the implicit expression of the electric field vector in cylindrical coordinates should generally have the following form:

$$\mathbf{E}(\rho, z) = E_\rho(\rho, z)\,\boldsymbol{u}_\rho + E_z(\rho, z)\,\boldsymbol{u}_z \tag{19}$$

Maxwell's second equation of the electrostatic field at point P in space is: curl $\mathbf{E} = \mathbf{0}$. Thus, the electric field vector can be derived from a scalar electric potential field by applying the gradient operator to the latter as follows: $\mathbf{E} = -\,\mathbf{grad}\,V$. Otherwise,

$$\mathbf{E} = -\nabla V \tag{20}$$

In cylindrical coordinates, the gradient operator vector in differential (20) is applied to the scalar function of the potential results as follows :

$$\mathbf{E}(\rho, z) = -\nabla V(\rho, z) = -\left(\frac{\partial}{\partial \rho}\boldsymbol{u}_\rho + \frac{\partial}{\partial z}\boldsymbol{u}_z\right) V(\rho, z) \tag{21}$$

The component of the electric field vector along the direction of the unit vector $\boldsymbol{u}_\phi$ is zero, i.e., $E_\phi = 0$, because the electric potential does not depend on the azimuthal angle ϕ in the invariance study. Nevertheless, the determination of the components of the field vector in (21) requires the derivation of the complete elliptic integral of the first kind, $K(k^2)$, into (18). This involves deriving function (15) as follows:

$$\frac{\partial K(k^2)}{\partial k} = \int_0^{\pi/2} \frac{\partial}{\partial k}\left(\frac{1}{\sqrt{1 - k^2 sin^2\beta}} d\beta\right) \tag{22}$$

After deriving, we obtain :

$$\frac{\partial K(k^2)}{\partial k} = \int_0^{\pi/2} \frac{k \sin^2\beta}{1 - k^2 sin^2\beta} \frac{d\beta}{\sqrt{1 - k^2 sin^2\beta}} \tag{23}$$

It is also easy to demonstrate that

$$\frac{\partial K(k^2)}{\partial k} = \frac{E(k^2) - (1 - k^2)\,K(k^2)}{k(1 - k^2)} \tag{24}$$

where $K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kinds respectively.

2.2.1. Deriving the Axial Component of the Electric Field Vector

To derive the axial component of the electric field vector that appears in the form (19) from the gradient of the potential in (21), it is necessary to obtain the following partial derivatives:

$$\frac{\partial q}{\partial z} = 2z \tag{25a}$$

$$\frac{\partial \sqrt{q}}{\partial z} = \frac{z}{\sqrt{q}} \tag{25b}$$

$$\frac{\partial K(k^2)}{\partial z} = \frac{\partial K(k^2)}{\partial k}\frac{\partial k}{\partial z} \tag{25c}$$

$$\frac{\partial k}{\partial z} = 2(\rho R)^{1/2}(2z)\left(-\frac{1}{2}\right)(q)^{-3/2} = -2z(\rho R)^{1/2}(q)^{-3/2}$$

$$= -\frac{zk}{q} = -\frac{zk^3}{4\rho R} \tag{25d}$$

$$\frac{\partial V(q, k^2)}{\partial z} = k_e Q \frac{2}{\pi} \frac{\frac{\partial K(k^2)}{\partial k}\frac{\partial k}{\partial z}\sqrt{q} - K(k^2)\frac{\partial \sqrt{q}}{\partial z}}{q} \tag{26}$$

Substituting derivatives (23) and (25a)–(25d) into (26), and after some simplifications, we obtain:

$$\frac{\partial V(q, k^2)}{\partial z} = -k_e \frac{Q}{\pi}\frac{2}{q^{3/2}} z \int_0^{\pi/2} \frac{1}{1 - k^2 sin^2\beta}\frac{d\beta}{\sqrt{1 - k^2 sin^2\beta}} \tag{27}$$

This is the expression of the axial component (E_z) of the electric field vector that appears in the gradient of the electric potential into (21) but with an extra minus (−) sign. Becaus the field vector derives from the potential, this negative sign should necessarily disappear after applying the axial component in the gradient function (21). Indeed, the axial component of the electric field vector is:

$$E_z(q, k^2) = -\frac{\partial V(q, k^2)}{\partial z} = k_e \frac{Q}{\pi}\frac{2}{q^{3/2}} z \int_0^{\pi/2} \frac{1}{1 - k^2 sin^2\beta}\frac{d\beta}{\sqrt{1 - k^2 sin^2\beta}} \tag{28}$$

Again, it is possible to stop at this stage and utilize a computer to numerically calculate this function. Alternatively, the calculations can be continued to establish the most compact possible expression.

Let us begin as follows. The complete elliptic integral of the third kind is defined as

$$\Pi(n|m) = \int_0^{\pi/2} \frac{1}{1 - n sin^2\beta}\frac{d\beta}{\sqrt{1 - m sin^2\beta}} \tag{29}$$

where n is the elliptic characteristic and m is the elliptic modulus [21]. When the parameter m is real, it can always be arranged as $0 \le m \le 1$ because $m = k^2$.

A noteworthy special case arises within this class of integral equations when both characteristic n and parameter m are equal to the modulus k squared. Specifically, when n equals m, i.e., when both are equal to k^2, the integral in (29) yields to

$$\Pi(k^2|k^2) = \int_0^{\pi/2} \frac{1}{1 - k^2 sin^2\beta}\frac{d\beta}{\sqrt{1 - k^2 sin^2\beta}} \tag{30}$$

According to Arfken and Weber [18] and Weisstein [21], when $k^2 = 1$, the function value is infinite. However, for $0 \le k^2 < 1$, the function is finite but increases as k^2 increases. Thus, $\Pi(k^2|k^2)$ diverges for $k^2 = 1$ and remains finite for $0 \le k^2 < 1$. For $k^2 = 0$, $\Pi(0|0) = \pi/2$.

Upon examining the axial component of the field given by function (28), it becomes apparent that it encompasses the specific case where $n = k^2$ and $m = k^2$ of the complete elliptic integral of the third kind. Consequently, the axial component of the electric field can be expressed as

$$E_z(q, k^2) = k_e \frac{Q}{\pi}\frac{2}{q^{3/2}} z\,\Pi(k^2|k^2) \tag{31}$$

Furthermore, a functional relation exists between the complete elliptic integral of the third kind, $\Pi(k^2|k^2)$, and the complete elliptic integral of the second kind, $E(k^2)$, as follows :

$$\Pi\left(k^2\left|k^2\right.\right)=\frac{E\left(k^2\right)}{1-k^2} \tag{32}$$

where

$$E\left(k^2\right)=\int_0^{\pi/2}\sqrt{1-k^2sin^2\beta}\,d\beta \tag{33}$$

Again, this integral also can be written as a power series [19]:

$$E\left(k^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)\,!}{2^{2n}\,(n!)^2}\right)^2\frac{k^{2n}}{1-2n}, \tag{34}$$

which is equivalent to

$$E\left(k^2\right)=\frac{\pi}{2}\left(1-\left(\frac{1}{2}\right)^2\frac{k^2}{1}-\left(\frac{1.3}{2.4}\right)^2\frac{k^4}{3}-\ldots-\left(\frac{(2n-1)\,!\,!}{(2n)\,!\,!}\right)^2\frac{k^{2n}}{2n-1}-\ldots\right) \tag{35}$$

If $k=0$, then $E(0)=\pi/2$, and if $k=1$, then $E(1)=1$.

For k^2=0, $E(0)=\pi/2$, and thus $\Pi(k^2|k^2)=E(0)=\pi/2$. For $k^2=1$, (1-k^2) = 0 in the denominator of (32) result in a diverging integral, indicating a singularity. However, this relationship remains valid and provides a method to calculate $\Pi(k^2|k^2)$ using $E(k^2)$. Subsequently, by replacing $\Pi(k^2|k^2)$ with its expression given by (32) into function (31), the more compact form of the axial component of the field vector is

$$E_z\left(q,k^2\right)=k_e\frac{Q}{\pi}\frac{2}{\left(1-k^2\right)q^{3/2}}z\,E\left(k^2\right) \tag{36}$$

This is the compact expression of the (Ez) axial component predicted by implicit function (19).

According to Mandre [5], this expression can be obtained by the same reasoning as above but using the differential equation for the complete elliptic integral of the first kind shown in function (24) instead of function (23) into the partial derivative (26) above. The demonstration is left to the reader as an exercise.

2.2.2. *Deriving the Radial Component of the Electric Field Vector*

The same reasoning used to derive the (E_z) axial component from the electric potential in cylindrical coordinates also applies to the (E_ρ) radial component. Similarly, E_ρ can be reduced to standard elliptic integral functions. To explicit the radial component of the electric field vector that appears in the form of (19) from the gradient of the potential in (21), we obtain the following partial derivatives:

$$\frac{\partial q}{\partial \rho}=2\left(\rho+R\right) \tag{37a}$$

$$\frac{\partial\sqrt{q}}{\partial\rho}==\frac{\rho+R}{\sqrt{q}} \tag{37b}$$

$$\frac{\partial K\left(k^2\right)}{\partial\rho}=\frac{\partial K\left(k^2\right)}{\partial k}\frac{\partial k}{\partial\rho} \tag{37c}$$

$$\begin{aligned}\frac{\partial k}{\partial\rho}&=4R\left(\frac{1}{2}\right)(4\rho R)^{-1/2}q^{-1/2}\\&+(4\rho R)^{1/2}\left(-\frac{1}{2}\right)q^{-3/2}\left(2R+2\rho\right)\\&=2R\,(4\rho R)^{-1/2}\frac{k}{2\,(\rho R)^{1/2}}\\&-(4\rho R)^{1/2}\left(\frac{1}{2}\right)\frac{k^3}{8\rho R\,(\rho R)^{1/2}}\left(2R+2\rho\right)\\&=\frac{k}{2\rho}-\frac{k^3}{8\rho R}\left(2R+2\rho\right)\\&=\frac{k}{2\rho}-\frac{k^3}{4\rho}-\frac{k^3}{4R}\end{aligned} \tag{37d}$$

Deriving function (18) again yields to

$$\frac{\partial V\left(q,k^2\right)}{\partial\rho}=k_eQ\frac{2}{\pi}\frac{\frac{\partial K\left(k^2\right)}{\partial k}\frac{\partial k}{\partial\rho}\sqrt{q}-K\left(k^2\right)\frac{\partial\sqrt{q}}{\partial\rho}}{q} \tag{38}$$

Substituting the partial derivatives (23) and (37a)–(37d) into (38), after some manipulations and simplifications, we obtain:

$$\frac{\partial V\left(q,k^2\right)}{\partial\rho}=-k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left[\rho\int_0^{\pi/2}\frac{1}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta+R\int_0^{\pi/2}\frac{1-2sin^2\beta}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\right] \tag{39}$$

From (21), the explicit form of the radial component of the electric field vector is

$$\begin{aligned}E_\rho\left(q,k^2\right)&=-\frac{\partial V\left(q,k^2\right)}{\partial\rho}\\&=k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left\{\rho\int_0^{\pi/2}\frac{1}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\right.\\&\left.+R\int_0^{\pi/2}\frac{1-2sin^2\beta}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\right\}\end{aligned} \tag{40}$$

Similarly, component $E\rho$ can be reduced to standard elliptic integral functions. Equation (40) shows that the radial component of the field contains two integral functions. The first integral between brackets is the complete elliptic integral of the third kind with $n=k^2$ and $m=k^2$, the second integral contains the term (1 $-2\sin^2\beta$) in the numerator, which makes it more complex. To simplify this, let us first break it down as follows:

$$E_\rho\left(q,k^2\right)=k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left\{\rho\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}}\right.$$

$$+R\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}} + \int_0^{\pi/2}\frac{-2Rsin^2\beta}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\Bigg\} \quad (41)$$

By multiplying the numerator and denominator of the third integral by ρ and rearranging, (41) can be rewritten as :

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\Bigg\{(\rho+R)\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}} + \frac{1}{\rho}\int_0^{\pi/2}\frac{-2R\rho sin^2\beta}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\Bigg\} \quad (42)$$

At this stage, it is necessary to add and subtract intruder terms to resolve the problem.

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\Bigg\{(\rho+R)\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}} + \frac{1}{\rho}\int_0^{\pi/2}\frac{\frac{R^2}{2}-\frac{R^2}{2}-2R\rho sin^2\beta+\frac{\rho^2}{2}-\frac{\rho^2}{2}+\frac{z^2}{2}-\frac{z^2}{2}}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\Bigg\} \quad (43)$$

Rearranging, the radial component yields

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left\{\begin{array}{l}(\rho+R)\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}} \\ +\frac{1}{\rho}\int_0^{\pi/2}\frac{\frac{1}{2}\left((R+\rho)^2+z^2\right)-2R\rho\, sin^2\beta-\frac{1}{2}\left(R^2+\rho^2+z^2+2R\rho\right)}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\end{array}\right\} \quad (44)$$

Using (13) to replace $q = [(\rho+R)^2+z^2]$ with $q = 4\rho R/k^2$, we obtain the following :

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left\{\begin{array}{l}(\rho+R)\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}} \\ +\frac{1}{\rho}\int_0^{\pi/2}\frac{\frac{2\rho R}{k^2}\left(1-k^2sin^2\beta\right)}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta-\frac{1}{\rho}\int_0^{\pi/2}\frac{\frac{1}{2}\frac{4\rho R}{k^2}}{\left(1-k^2sin^2\beta\right)^{3/2}}d\beta\end{array}\right\} \quad (45)$$

A more convenient form can be obtained by making the appropriate simplifications. The novel form is:

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\Bigg\{(\rho+R)\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}} + \frac{2R}{k^2}\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{1/2}} - \frac{2R}{k^2}\int_0^{\pi/2}\frac{d\beta}{\left(1-k^2sin^2\beta\right)^{3/2}}\Bigg\} \quad (46)$$

The middle integral is the complete elliptic integral of the first kind: $K(k^2)$, already given by (15). Additionally, the first and third integrals in function (46) are special cases of the complete elliptic integral of the third kind $\Pi(n|m)$ with $n = k^2$ and $m = k^2$. This is given by (30) above.

By replacing, the expression of the radial component can be rewritten as :

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left\{(\rho+R)\,\Pi\left(k^2\left|k^2\right.\right)+\frac{2R}{k^2}K\left(k^2\right) - \frac{2R}{k^2}\Pi\left(k^2\left|k^2\right.\right)\right\} \quad (47)$$

Then, using (32) again, we obtain :

$$E_\rho\left(q,k^2\right) = k_e\frac{Q}{\pi}\frac{2}{q^{3/2}}\left\{(\rho+R)\frac{E\left(k^2\right)}{1-k^2}+\frac{2R}{k^2}K\left(k^2\right) - \frac{2R}{k^2}\frac{E\left(k^2\right)}{1-k^2}\right\} \quad (48)$$

At this stage, expression was reduced as much as possible. The final step is arranging it in a different order, as follows :

$$E_\rho\left(q,k^2\right) = k_eQ\frac{2}{\pi k^2\left(1-k^2\right)q^{3/2}} \left\{2R\left(1-k^2\right)K\left(k^2\right)-\left(2R-k^2\left(\rho+R\right)\right)E\left(k^2\right)\right\} \quad (49)$$

This is the expression of the radial component (E_ρ) predicted by the implicit function (19).

According to Mandre [5], this expression can be obtained by following the same reasoning as above, but using the differential equation for the complete elliptic integral of the first kind shown in function (24) instead of function (23) into the partial derivative (38). Once again, the demonstration is left to the reader.

3. Results

Consequently, we arrive at explicit expressions for the electrostatic field components in (19) as in (21). The radial and axial components of the electric field vector of a uniformly charged ring at any point in space are as follows:

$$E_\rho\left(q,\mu\right) = k_eQ\frac{2}{\pi\mu\left(1-\mu\right)q^{3/2}} \left(2R\left(1-\mu\right)K\left(\mu\right)-\left(2R-\mu\left(\rho+R\right)\right)E\left(\mu\right)\right) \quad (50a)$$

$$E_z\left(q,\mu\right) = k_e\frac{Q}{\pi}\frac{2}{\left(1-\mu\right)q^{3/2}}z\,E\left(\mu\right) \quad (50b)$$

where

$$q = \rho^2+R^2+z^2+2\rho R,$$

$$\mu = k^2 = \frac{4\rho R}{q},$$

$K(\mu)$ and $E(\mu)$ are the complete elliptic integrals of the first and second kinds, respetively.

These are the most compact expressions for the axial and radial components of the electric field vector of a uniformly charged ring. Point $k^2 = 1$ is a point of singularity for the curves of the complete elliptic integrals of the first and third kinds. Consequently, the radial component should have this point as a singularity point. This causes the radial component to tend towards infinity when $k \to = 1$. The calculations of the complete elliptic integrals into the components (50a) and (50b) can be performed using software such as MATLAB, Mathematica, and ALGLIB Elliptic Integral subroutines for C++/Java/Python.

The previous calculations are valid assuming that the ring is infinitely thin. But does such a ring exist? This is why it is important to remember that real conductors—such us rings—have a certain volume. They are physical bodies. In this case and accordingly to the laws of Electrostatics, the field inside the volume of the ring must be zero because its surface is closed and any excess charge exits to the surface.

4. In-Plane and On-Axis Electric Field Vector

4.1. Electric Field in the Plane of the Ring

The condition $z = 0$ (i.e., $\theta = \pi/2$) relocates point P to the xy plane, which is the plane of the ring, because P $\in xy$ plane, see Fig. 1. This means that the axial component (E_z) in (50b) is zero, thereby confirming that the electric field of the ring is purely radial as expected from (19) by symmetry.

Thus, the non-zero component of the field is:

$$E_r(r,0) = k_e \frac{Q}{\pi} \frac{2}{q^{3/2}} \frac{1}{\mu} \frac{1}{(1-\mu)} \{2R(1-\mu)K(\mu) - (2R - \mu(r+R))\ E(\mu)\} \tag{51}$$

In order to simplify the calculation of function (51) on a computer, we can set $a = \rho/R$ and $b = z/R$, where ρ and z are given by (4). When $\theta = \pi/2$ (i.e., when $\cos(\theta) = 0$ and $\sin(\theta) = 1$), it can be deduced that $a = \rho/R = r/R$ and $b = 0$, meaning that the parameters in (13) simplify to

$$q = R^2 + r^2 + 2rR = R^2(1 + 2a + a^2) = R^2(1+a)^2 \tag{52a}$$

and

$$\mu = k^2 = \frac{4rR}{q} = \frac{4rR}{(R+r)^2} = \frac{4a}{1+2a+a^2} \tag{52b}$$

Since $Q = 2\pi R\lambda$, function (51) can be simplified as follows:

$$E_r(r,0) = k_e \frac{\lambda}{R} \frac{(1+a)}{a} \frac{1}{(1-a)^2} \{2(1-\mu)K(\mu) - (2 - \mu(1+a))\ E(\mu)\} \tag{53}$$

This function can be integrated using a computer. Function (53) has two variables: the angle α and the parameter $a = r/R$. The integration calculation must be performed on these variables.

As shown in Table I, the field values are presented in terms of $k_e\lambda/R$ in SI units for various concentric plane circles with relative radii ($a = r/R$) between 0 and 4 ($a \in [0, 1[\cup]1, 4]$). As previously mentioned, it is evident that the field in the plane of the ring diverges at $r/R = 1$, leading to a singularity.

TABLE I: The In-plane Value of the Field of a Uniform Ring of Charge is Shown as a Function of the Relative Radius, $a = r/R$

$a = r/R$	E_r ($\times k_e\lambda/R$)	$a = r/R$	E_r ($\times k_e\lambda/R$)
0	0,000	1,001	46,188
0,1	−0,318	1,1	22,005
0,2	−0,658	1,2	11,310
0,3	−1,048	1,3	7,551
0,4	−1,528	1,4	5,606
0,5	−2,167	1,5	4,410
0,6	−3,100	1,6	3,599
0,7	−4,641	1,7	3,015
0,8	−7,744	1,8	2,574
0,9	−17,184	1,9	2,231
0,999	−32,368	2	1,957

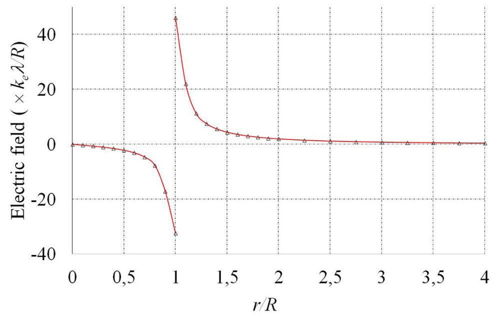

Fig. 2. The plot of the electric field amplitude versus the relative radius $a = r/R$ in the plane of the ring for λ>0; the field vanishes at the center of the ring. When $r/R \to 1$, $E_r \to \infty$: the field diverges.

The data in Table I can be interpreted as follows: for example, at a circle of relative radius $a = r/R = 0.5$, the electric field vector is approximately:

$$\mathbf{E} \cong -2,167 \frac{k_e\lambda}{R} \boldsymbol{u}_r \tag{54}$$

The minus sign (−) in this expression indicates that the vector's direction is currently opposite to the unit vector $\boldsymbol{u}_r$, this is because $\lambda > 0$. Outside the ring, the vector is oriented in the same direction as $\boldsymbol{u}_r$. In general, on a given field line, the orientation of the electric field vector outside the area delimited by the ring is opposite to its orientation inside the area.

Fig. 2 shows the plot of the field in the plane of the ring as a function of the ratio $a = r/R$, as given by (53). Clearly, the field is zero at the centre of the ring and diverges at $a = 1$, a singular point marking the edge of the ring. As point P approaches the ring's edge, the field tends towards infinity in a manner similar to the behaviour of elliptic integrals in expression (15) of $K(\mu)$.

4.2. The Electric Field on the z Axis

By symmetry, only the axial component of the field is non-zero on the z axis; the radial component is zero. The field along the z axis of the ring corresponds to $\theta = 0$, as shown in Fig. 1. This condition means that the point P is on the z axis, i.e., $r = z$ and $b = z/R$, because P is on the z axis. If $\theta = 0$, then $\cos(\theta) = 1$ and $b = z/R = r/R$. However, on the z axis, the r spherical coordinate coincides with the z coordinate, and $\sin(\theta) = 0$, so $a = 0$.

Now, by substituting the parameters ρ, z, q and k, by their respective values, assuming that $\rho = 0$, $q = (R^2 + z^2)$, $z = r$, $\mu = k^2 = 0$ in (50a) and (50b), and simplifying the resulting expressions, we obtain

$$E_\rho\,(q, 0) = 0 \tag{55}$$

and

$$E_z\,(q, 0) = k_e \frac{Qz}{\left(R^2 + z^2\right)^{3/2}} z \tag{56}$$

This is because $K(0) = \pi/2$ at $\mu = k^2 = 0$.

The coordinate z represents the distance from the centre O of the ring to point P on the z axis.

The total electric field vector is

$$\mathbf{E} = k_e \frac{Qz}{\left(R^2 + z^2\right)^{3/2}} \boldsymbol{u}_z \tag{57}$$

Most undergraduate students are very familiar with this formula, as it is the only one found in many textbooks and websites on Electrostatics [One, p. 14], [2, p. 707], [3, p. 697].

However, if P is at the center, $z = 0$ then $\mathbf{E} = \mathbf{0}$.

When $z >> R$, vector (57) approximately gives

$$\mathbf{E} = k_e \frac{Qz}{\left(R^2 + z^2\right)^{3/2}} \boldsymbol{u}_z \approx k_e \frac{Q}{z^2} \boldsymbol{u}_z = k_e \frac{Q}{r^2} \boldsymbol{u}_z \tag{58}$$

If $z >> r$, the uniform ring of charge Q acts as a point charge for locations far from the ring.

Now suppose that a test charge q_t is placed at the centre of the uniform ring of charge and shifted slightly by a distance $z << R$ along the z axis. The expression for the force acting on q_t is:

$$\mathbf{F} = q_t\,\mathbf{E} = k_e \frac{Qq_t z}{R^3} \boldsymbol{u}_z \tag{59}$$

Thus, when the charge q_t is released, it oscillates at a frequency of

$$f = \frac{1}{2\pi} \sqrt{k_e \frac{q_t Q}{mR^3}} \tag{60}$$

The force that acts on q_t has the form of Hooke's law [2, p. 707], like for the spring.

5. The Field Along an Axis Parallel to the z Axis of the Ring

The obtained results allow us to calculate the field values along an axis parallel to the ring's axis without the need for laborious calculations, such as those described by Escalante [8]. In fact, by implementing functions (50a) and (50b), the field values at any point in space can be calculated using a computer. For example, one can force the parameter $a = \rho/R$ to be *const*. In the xy plane, regardless of the values of the r and θ coordinates. If $a = \rho/R$ is fixed (e.g., $a = 0.5$), varying the extremity of the position vector $\boldsymbol{r}$ on an axis parallel to the z axis causes the angle θ to vary, forcing the point P to move along this axis, which is parallel to the z axis and perpendicular to the xy plane, through the extremity of the position vector at a level whose projection onto the xy plane is the point $a =$ const. These calculations can be performed numerically using computer software.

6. Discussion

The obtained results show that the field of a uniformly charged ring is not uniform in a 3D-space. Even so, the motion of the initially static charged particles follows trajectories that are tangent to the field lines at each point of space. Electric fields are advantageous because they are widely used in devices designed for scientific and engineering applications, such as electrostatic ion storage rings and electrostatic ion beam traps as asserted by [22] and [23]. Charged rings are also used in mass spectrometers [4] and in electronic optics [24]. Moreover, the rings of charge help confine plasma using electromagnetic forces in nuclear fusion research such as Tokamak Fusion Reactors.

Nevertheless, the shapes and sizes of the rings used in electrostatic devices vary from one application to another. In particular, the charge and size of microscopic rings can only be measured using atomic force microscope [10]. A major challenge is the problem of calculating tip-sample forces in atomic force microscopy [9], [11]. A challenge of electric field and force calculations for measuring the charge of rings using the scanning force microscopy technique was reported by Gordon et al. [10]. Particle-beam weapons [25], oscillations of a pair of charged rings [26], etc., constitute endless research topics. Also, it is necessary to consider optimising calculations for high numerical accuracy for applications in nanophysics.

7. Conclusion

The analytical expression of the scalar function of the electric potential field is derived using Coulomb's law. However, by pushing the calculations to their limits, the most compact possible expression was obtained. The latter expression is primarily a function of elliptic integrals of the first kind, as well as certain parameters associated with the uniform ring of charge. The expression for the electric field vector is then obtained from the gradient of the scalar function of the electric potential. Applying the gradient operator revealed that the radial and axial components are non-zero and that the azimuthal component of the electric field is zero. By pushing the calculations to their limits, the most compact possible expressions for the radial and axial components are obtained using two methods. The first method uses the derivative of the complete elliptic integral of the first kind, and the second uses a differential equation for the complete elliptic integral of the first kind, which is tabulated in several reference books on special functions, such as elliptic integrals, as well as on websites. It was found that the radial and axial components of the field are primarily functions of elliptic integrals of the first and second kinds, as well as certain parameters associated with the ring of charge. This approach is a valuable pedagogical tool that enables students to explore and comprehend special functions in the context of electromagnetism courses. For instance, it can be used in computer-based learning environments such as computer software. This task is left to be carried out by any curious reader.

The study offers valuable implications and future directions. The compact expressions are useful for computational implementations and advanced studies in Electrostatics. While the calculations presented herein were limited to infinitely thin rings, the results can be used to determine the field of rings of various shapes and finite thicknesses with almost any charge distribution. From this perspective, the shift from a single ring to multiple coaxial rings (disk modeling) is a possibility. Because the principles of superposition apply, it is only necessary to approximate any odd-shaped ring using a certain number of thin-walled rings and then add together the resulting fields. The accuracy of the solution depended on the number of concentric rings used to approximate the actual shape. In this study, the non-zero components of the electric field were determined as a function of elliptic integrals in cylindrical coordinates. Generalization enables us to extend our understanding beyond ideal rings, non-uniform charge densities, deformed geometries and multi-ring systems. As an extension, it is also possible to calculate the same components in Cartesian and spherical coordinates.

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Conflict of Interest

The author declares that he has no conflicts of interest.

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